Geometric Complexity Theory VIII: On canonical bases for the nonstandard quantum groups

(extended abstract)

Dedicated to Sri Ramakrishna

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Abstract

This article gives conjecturally correct algorithms to construct canonical bases of the irreducible polynomial representations and the matrix coordinate rings of the nonstandard quantum groups in GCT4 and GCT7, and canonical bases of the dually paired nonstandard deformations of the symmetric group algebra therein. These are generalizations of the canonical bases of the irreducible polynomial representations and the matrix coordinate ring of the standard quantum group, as constructed by Kashiwara and Lusztig, and the Kazhdan-Lusztig basis of the Hecke algebra. A positive ($\#P$-) formula for the well-known plethysm constants follows from their conjectural properties and the duality and reciprocity conjectures in [GCT7].

1 Introduction

Let H be a complex connected classical reductive group, $X = V_\mu(H)$ its irreducible polynomial representation with highest weight μ, $G = GL(X)$, and $\rho : H \to G$ the representation map. Given a highest weight π of H and λ of G, the plethysm constant $a^\pi_{\lambda,\mu}$ is defined to be the multiplicity of $V_\pi(H)$ in $V_\lambda(G)$, considered an H-module via ρ. A fundamental problem in representation theory is to find a positive ($\#P$-) formula (rule) for the plethysm constant [GCT7, St] akin to the Littlewood-Richardson rule. Motivated by this problem, the article [GCT7] constructs a quantization ρ_q of the homomorphism ρ in the form

$$\rho_q : H_q \to G_q^H, \tag{1}$$

where H_q is the standard (Drinfeld-Jimbo) quantum group [Dri, Ji, RTF] associated with H and G_q^H is the new (possibly singular) quantum group, called *the nonstandard quantum group* associated with ρ. In the standard case, i.e., when $H = G$, this specializes to the standard quantum group, and in the Kronecker case, i.e., when $H = GL(V) \times GL(W)$, $X = V \otimes W$ with the natural H action, this specializes to the nonstandard quantum group in [GCT4]. Also constructed in [GCT7] is a nonstandard quantization $\mathcal{B}_r^H(q)$ of the group algebra $\mathbb{C}[S_r]$, S_r the symmetric group, whose relationship with G_q^H is conjecturally similar to that of the Hecke algebra with the standard quantum group.

This article gives conjecturally correct algorithms for constructing canonical bases of the irreducible polynomial representations and the matrix coordinate ring of G_q^H (Section 2) and a canonical basis of $\mathcal{B}_r^H(q)$ (Section 3). We call these *nonstandard canonical bases*. They are generalizations of the canonical bases of the irreducible polynomial representations and the matrix coordinate ring of the standard quantum group, as constructed by Kashiwara and Lusztig [Kas1, Kas2, Lu1, Lu2], and the Kazhdan-Lusztig basis [KL1] of the Hecke algebra. A positive ($\#P$-) formula for the plethysm constant follows from their conjectural properties (Sections 2.4 and 3.4), which are akin to those of the standard canonical basis, and the conjectural duality and reciprocity between G_q^H and $\mathcal{B}_r^H(q)$; cf. [GCT7].

Experimental evidence (Section 5) suggests that these algorithms should be correct. But we can not prove this formally, nor the required properties of the nonstandard canonical bases. Mainly because we are unable to deal with the complexity of the minors of the nonstandard quantum group. Specifically, in contrast to the elementary formula for the Laplace expansion of a minor of the standard quantum group–which is akin to the classical Laplace expansion at $q = 1$–the Laplace expansion of a minor of a nonstandard quantum group is highly nonelementary; cf. [GCT7]. Its coefficients depend on the multiplicative structural constants of a canonical basis akin to the canonical basis of the coordinate ring of the standard quantum group as per Kashiwara and Lusztig. In the Kronecker case, these constants are conjecturally polynomials in $Q[q, q^{-1}]$ with nonnegative coefficients, and in general, polynomials with a conjectural relaxed form of this property. To prove this and to get explicit formulae for the minors in the nonstandard setting, one needs explicit interpretations for these structural constants in the spirit of the interpretation based on perverse sheaves for the Kazhdan-Lusztig polynomials [KL2] and the multiplicative structural constants of the canonical basis of the Drinfeld-Jimbo enveloping algebra [Lu2]. Thus even to get explicit formulae for the minors of the nonstandard quantum group a (nonstandard) extension of the theory of perverse sheaves [BBD], and the underlying Riemann hypothesis over finite fields [Dl2] seems necessary.

Minors of the standard quantum group are in a sense the simplest (basic) canonical basis elements in its matrix coordinate ring. That the simplest canonical basis elements for the nonstandard quantum group–namely, its minors–are already so nonelementary in contrast to the standard case indicates the possible difficulties that may be encountered in proving correctness of the algorithms given here for constructing nonstandard canonical bases.

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Notation: We use the symbols π and μ to denote labels of irreducible representations of the standard quantum group and the symbols $\alpha, \alpha_0, \alpha_1, \ldots$ to denote labels of irreducible representations of the nonstandard quantum group. Thus objects with subscripts π and μ are standard and the objects with subscripts $\alpha, \alpha_0, \ldots$ are nonstandard.

2 Nonstandard canonical basis for G_q^H

In this section we describe a conjecturally correct algorithm for constructing the canonical basis of the matrix coordinate ring of the nonstandard quantum group G_q^H. We follow the same terminology as in [Kli] for the basic quantum group notions.

For the sake of simplicity, let us assume that $H = GL(V)$. Let $H_q =$

$GL_q(V)$ denote its standard (Drinfeld-Jimbo) quantization [Dri, Ji, RTF], and $M_q(V)$ the standard quantization of the matrix space $M(V)$. Let $\mathcal{O}(M_q(V))$ be the coordinate ring of $M_q(V)$. We call it *the matrix coordinate ring* of $GL_q(V)$. The coordinate ring $\mathcal{O}(GL_q(V))$ of $GL_q(V)$ is obtained by localizing $\mathcal{O}(M_q(V))$ at the quantum determinant of $GL_q(V)$. Let $\mathcal{H}$ denote the Lie algebra of H, and $U_q(\mathcal{H})$ the Drinfeld-Jimbo universal enveloping algebra of $H_q = GL_q(V)$.

To quantize the homomorphism $\rho : H \to G = GL(X)$ as in (1), the article [GCT7] constructs a *nonstandard matrix coordinate ring* $\mathcal{O}(M_q^H(X))$ of a (virtual) *nonstandard matrix space* $M_q^H(X)$, and then defines the *nonstandard quantized universal enveloping algebra* $U_q^H(\mathcal{G})$ by dualization. The *nonstandard quantum group* G_q^H is the virtual object whose universal enveloping algebra is $U_q^H(\mathcal{G})$. The construction also yields natural bialgebra homomorphisms from $U_q(\mathcal{H})$ to $U_q^H(\mathcal{G})$ and from $\mathcal{O}(M_q^H(X))$ to $\mathcal{O}(M_q(V)$, thereby giving the desired quantizations of the homomorphisms $U(\mathcal{H}) \to U(\mathcal{G})$ and $\mathcal{O}(M(X)) \to \mathcal{O}(M(V))$. This is what is meant by the quantization (1) of the representation map ρ. The determinant of G_q^H may vanish, and hence, we cannot, in general, define its coordinate ring $\mathcal{O}(G_q^H)$ by localizing $\mathcal{O}(M_q^H(X))$. Fortunately, this does not matter since $\mathcal{O}(M_q^H(X))$ still has properties similar to that of the standard matrix coordinate ring $\mathcal{O}(M_q(V))$. Specifically, it is cosemisimple. This means all (finite dimensional) *polynomial representations* of G_q^H, by which we mean corepresentations of $\mathcal{O}(M_q^H(V))$, are completely reducible. A nonstandard quantum analogue of the Peter-Weyl theorem holds: i.e.,

$$\mathcal{O}(M_q^H(X)) = \bigoplus_{\alpha} W_{q,\alpha}^* \otimes W_{q,\alpha}, \tag{2}$$

where $W_{q,\alpha}$ runs over all irreducible corepresentations of $\mathcal{O}(M_q^H(X))$. Furthermore, the nonstandard enveloping algebra $U_q^H(\mathcal{G})$ is a bialgebra with a compact real form ($*$-structure).

The goal is to construct a canonical basis for the matrix coordinate ring $\mathcal{O}(M_q^H(X))$ akin to the canonical basis of the standard matrix coordinate ring $\mathcal{O}(M_q(V))$ as per Kashiwara and Lusztig [Kas1, Kas2, Lu1, Lu2].

2.1 The standard setting

We begin by reviewing the basic scheme of Kashiwara and Lusztig for constructing a canonical basis of the matrix coordinate ring $\mathcal{O}(M_q(V))$. The canonical basis of the coordinate ring $\mathcal{O}(GL_q(V))$ is obtained by localizing at the determinant.

Following Kashiwara, we first define a balanced triple. Let A and $\bar{A}$ be the ring of rational functions in q regular at $q = 0$ and $q = \infty$, respectively. Let V be a $Q(q)$-vector space, L_0 a sub-A-module (A-lattice) of V, L_∞ a sub-$\bar{A}$-module ($\bar{A}$-lattice) of V, and V_Q a sub-$Q[q, q^{-1}]$-module of V such that

$$V \cong Q(q) \otimes_{Q[q,q^{-1}]} V_Q \cong Q(q) \otimes_A L_0 \cong Q(q) \otimes_{\bar{A}} L_\infty.$$

We say that (V_Q, L_0, L_∞) is a *balanced triple* if any of the following three equivalent conditions hold:

(a) $E = V_Q \cap L_0 \cap L_\infty \to L_0/qL_0$ is an isomorphism.

(b) $E \to L_\infty/q^{-1}L_\infty$ is an isomorphism.

(c) $Q[q, q^{-1}] \otimes_Q E \to V_Q$, $A \otimes_Q E \to L_0$, $\bar{A} \otimes_Q E \to L_\infty$ are isomorphisms.

Let $R = \mathcal{O}(M_q(V))$. Kashiwara constructs an A-submodule (lattice) $L = L(R) \subset R$, an involution $-$ of R, a $Q[q, q^{-1}]$-submodule $R_Q \subset R$, and a basis B of L/qL such that $(R_Q, L, \bar{L})$ is a balanced triple and, letting G denote the inverse of the isomorphism $R_Q \cap L \cap \bar{L} \to L/qL$, $\{G(b) \mid b \in B\}$ is the canonical basis of R. The pair (L, B), called the (upper) crystal base of R, has the following form. By the q-analogue of the Peter-Weyl theorem for the standard quantum group,

$$R = \oplus_\pi V_{q,\pi}^* \otimes V_{q,\pi} \tag{3}$$

as a bi-$GL_q(V)$-module, where $V_{q,\pi} = V_{q,\pi}(V)$ is the irreducible polynomial representation of $GL_q(V)$ with highest weight π. Let (L_π, B_π) denote the upper crystal base of $V_{q,\pi}$. Then

$$(L, B) = \oplus_\pi (L_\pi^*, B_\pi^*) \otimes (L_\pi, B_\pi) \tag{4}$$

with appropriate normalization.

We now describe a construction of the upper crystal base (L_π, B_π) that can be generalized to the nonstandard setting.

Let r be the size of the partition π. Choose any embedding $\rho : V_{q,\pi} \to V^{\otimes r}$ such that the highest weight vector of the image $V_{q,\pi}^\rho = \rho(V_{q,\pi})$ belongs to the A-lattice $L(V)^{\otimes r}$ of $V^{\otimes r}$, where $L(V)$ denotes the lattice of V generated by its standard basis $\{v_i\}$. We also assume that the highest weight vector does not belong to $qL(V)^{\otimes r}$. Choose a Hermitian form on $V^{\otimes r}$ so that its monomial basis $\{v_{i_1} \otimes \cdots v_{i_r}\}$ is orthonormal. Let $V_{q,\pi}^{\rho,\perp}$ denote the orthogonal complement of $V_{q,\pi}^\rho$. Since $GL_q(V)$ has a compact real form $U_q(V)$–i.e., the unitary compact subgroup in the sense of Woronowicz [W]–it follows that $V_{q,\pi}^{\rho,\perp}$ is a $GL_q(V)$-module. Thus $V^{\otimes r} = V_{q,\pi}^{\rho} \oplus V_{q,\pi}^{\rho,\perp}$ as a $GL_q(V)$-module. Let

$$L_\pi^\rho = L(V)^{\otimes r} \cap V_{q,\pi}^\rho \quad \text{and} \quad L_\pi^{\rho,\perp} = L(V)^{\otimes r} \cap V_{q,\pi}^{\rho,\perp}.$$

It follows from Kashiwara's work [Kas1] that $L(V)^{\otimes r} = L_\pi^\rho \oplus L_\pi^{\rho,\perp}$.

Let $B(V) = \{b_i = \psi(v_i)\}$ denote the basis of $L(V)/qL(V)$, where $\psi : L(V) \to L(V)/qL(V)$ is the natural projection. Let $B(V)^{\otimes r} = \{b_{i_1} \otimes \cdots \otimes b_{i_r}\}$ denote the monomial basis of $B(V)^{\otimes r}$. Given $b \in B(V)^{\otimes r}$, let

$$b = \sum_{i_1,\ldots,i_r} f(b; i_1, \ldots, i_r) b_{i_1} \otimes \cdots \otimes b_{i_r},$$

be its expansion in the monomial basis. The set of monomials $b_{i_1} \otimes \cdots \otimes b_{i_r}$ such that the coefficients $f(b; i_1, \ldots, i_r)$ are nonzero is called the *monomial support* of b.

It follows from the works of Kashiwara [Kas1] and Date et al [DJM] that L_π^ρ/qL_π^ρ has a unique basis B_π^ρ (up to scaling by constant multiples) such that the monomial supports of its elements are disjoint–in fact, one can choose ρ so that the monomial support of each basis element consists of just one distinct monomial. This basis can be made completely unique by appropriate normalization. Then (L_π^ρ, B_π^ρ) coincides with the upper crystal base of $V_{q,\pi}$ as constructed by Kashiwara. Furthermore, this crystal base does not depend on the embedding ρ (up to isomorphism). Hence, we let $(L_\pi, B_\pi) = (L_\pi^\rho, B_\pi^\rho)$ for any ρ as above. Kashiwara [Kas1] also shows that L_π and $B_\pi \cup \{0\}$ are invariant under certain crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ corresponding to the simple roots of H_q.

This scheme for constructing the upper crystal base (L_π, B_π) crucially depends on the existence of a compact real form $U_q(V) \subseteq GL_q(V) = H_q$. Even the existence of a compact real form of the standard Drinfeld-Jimbo enveloping algebra $U_q(\mathcal{H})$ suffices here.

2.2 Nonstandard triple

We now generalize the preceding scheme to the nonstandard setting using the compact real form of $U_q^H(\mathcal{G})$, whose existence is proved in GCT7. The goal is to construct an analogous triple for the matrix coordinate ring $S = \mathcal{O}(M_q^H(X))$ of G_q^H. It will turn out that this triple need not be balanced as in the standard case. We shall describe in Section 2.3 how a canonical basis can be constructed from such a triple despite the lack of balance.

We begin by recalling that the q-analogue of the Peter-Weyl theorem (2) in the nonstandard setting need not hold over $Q(q)$ unlike in the standard setting. It holds only over an appropriate algebraic extension K of $Q(q)$ [GCT7]–thinking of q as a transcendental. It will be convenient to assume in what follows that K is actually an algebraic extension $\tilde{Q}(q)$, where $\tilde{Q}$ is the algebraic closure of Q. We let A_K and $\bar{A}_K$ be the subrings of algebraic functions in K that are regular at $q = 0$ and $q = \infty$, respectively. Let K_Q be the integral closure of $Q[q, q^{-1}]$ in K. Clearly, $K_Q \cap A_K \cap \bar{A}_K = \hat{Q}$, where $\hat{Q}$ denotes the integral closure of Q in $\tilde{Q}$. In what follows, we let $\hat{Q}, A_K, \bar{A}_K$ and K_Q play the role of $Q, A, \bar{A}$ and $Q[q, q^{-1}]$ in Section 2.1. Thus by a lattice at $q = 0$ we mean an A_K-lattice, by a lattice at $q = \infty$ a $\bar{A}_K$-lattice, by a Q-form, a K_Q-form (module). Similarly, instead of Q-modules and $Q(q)$-modules, we will be considering $\hat{Q}$-modules and K-modules. That is, in what follows we shall assuming that the underlying base field is K, instead of $Q(q)$.

Now let us describe the construction of the upper crystal base of an irreducible polynomial representation $W_{q,\alpha}$ of G_q^H. Let $X_q = V_{q,\mu}$ denote the quantization of $X = V_\mu$; i.e, the H_q-module with highest weight μ. Since the underlying field is K, X_q is a K-module. Let $\{z_i\}$ denote its standard q-orthonormal Gelfand-Tsetlin basis. Let $\{x_i\}$ denote the rescaled version of Gelfand-Tsetlin basis (as described in Section 7.3.3 of [Kli]) so that there are no square roots in the explicit formulae for the action of the generators of the Drinfeld-Jimbo algebra $U_q\mathcal{H}$ on this basis. Alternatively, we can let $\{x_i\}$ be the standard (upper) canonical basis [Kas1, Lu2] of X_q as an H_q-module. We assume that $\lim_{q\to 0}(z_i - x_i) = 0$; this can always be arranged. In what follows, we sometimes denote X_q by X. What is meant should be clear from the context. It is shown in [GCT7] that $W_{q,\alpha}$ can be embedded in $X^{\otimes r}$ for an appropriate r. We choose an embedding $\rho : W_{q,\alpha} \to X^{\otimes r}$ as follows.

If $r = 1$, $W_{q,\alpha} = X$, so this is trivial. Otherwise, choose any β of degree $r-1$ so that $W_{q,\alpha}$ occurs as a G_q^H-submodule of $W_{q,\beta} \otimes X$. By semisimplicity, the latter is completely reducible as a G_q^H-module:

$$W_{q,\beta} \otimes X = \oplus_{\beta_j} W_{q,\beta_j}, \tag{5}$$

where we assume that this is also a decomposition as a K-module. Choose any j such that $W_{q,\beta_j} \cong W_{q,\alpha}$. This fixes an embedding of $W_{q,\alpha}$ in $W_{q,\beta} \otimes X$.

(It is a plausible conjecture that the decomposition (5) is multiplicity free. This would be a conjectural analogue of Pieri's rule in the nonstandard setting. It would imply that the embedding of $W_{q,\alpha}$ in $W_{q,\beta} \otimes X$ is unique. But this is not required here.). By induction, we have fixed an embedding of $W_{q,\beta}$ in $X^{\otimes r-1}$. This fixes an embedding ρ of $W_{q,\alpha}$ in $X^{\otimes r}$ (among many possible choices). Let $W^{\rho}_{q,\alpha} = \rho(W_{q,\alpha})$ be its image.

Choose a Hermitian form on $X^{\otimes r}$ so that its Gelfand-Tsetlin basis $\{z_{i_1} \otimes \cdots \otimes z_{i_r}\}$ is orthonormal. Let $W^{\rho,\perp}_{q,\alpha}$ denote the orthogonal complement of $W^{\rho}_{q,\alpha}$. Since $U^H_q(\mathcal{G})$ has a compact real form [GCT7] such that $X^{\otimes r}_q$ is its unitary representation with respect to this Hermitian form, it follows that $W^{\rho,\perp}_{q,\alpha}$ is a G^H_q-module. Thus $X^{\otimes r} = W^{\rho}_{q,\alpha} \oplus W^{\rho,\perp}_{q,\alpha}$ as a G^H_q-module. Let

$$L^{\rho}_{\alpha} = L(X)^{\otimes r} \cap W^{\rho}_{q,\alpha} \quad \text{and} \quad L^{\rho,\perp}_{\alpha} = L(X)^{\otimes r} \cap W^{\rho,\perp}_{q,\alpha}.$$

Then, in analogy with Kashiwara's work mentioned above:

Proposition 2.1 $L(X)^{\otimes r} = L^{\rho}_{\alpha} \oplus L^{\rho,\perp}_{\alpha}$.

Proof: The r.h.s. is clearly contained in the l.h.s. To show the converse it suffices to show that L^{ρ}_{α} and $L^{\rho,\perp}_{\alpha}$ are projections of the lattice $L(X)^{\otimes r}$ onto $W^{\rho}_{q,\alpha}$ and $W^{\rho,\perp}_{q,\alpha}$ respectively. Let us show this for L^{ρ}_{α}, the other case being similar. Clearly, the projection of $L(X)^{\otimes r}$ onto $W^{\rho}_{q,\alpha}$ contains L^{ρ}_{α}. We only have to show that the projection $\hat{y}$ of any $y \in L(X)^{\otimes r}$ onto $W^{\rho}_{q,\alpha}$ also belongs to the lattice $L(X)^{\otimes r}$, and hence to L^{ρ}_{α}. Since $y \in L(X)^{\otimes r}$, its length $|y|$ w.r.t. the preceding Hermitian form tends to a well defined nonnegative real number as $q \to 0$. Since, the projection $y \to \hat{y}$ is orthonormal, the length $|\hat{y}|$ of $\hat{y}$ is at most $|y|$, and hence also tends to a well defined nonnegative real number as $q \to 0$. This means $\hat{y}$ is regular at $q = 0$ and hence belongs to $L(X)^{\otimes r}$. Q.E.D.

Let $B(X) = \{b_i = \phi(x_i)\}$ denote the basis of the $\tilde{Q}$-module $L(X)/qL(X)$, where $\phi : L(X) \to L(X)/qL(X)$ is the natural projection (the b_i's in this section are different from the b_i's in Section 2.1). Let $B(X)^{\otimes r} = \{b_{i_1} \otimes \cdots \otimes b_{i_r}\}$ denote the monomial basis of $L(X)^{\otimes r}$. Given $b \in B(X)^{\otimes r}$, let

$$b = \sum_{i_1,\ldots,i_r} g(b; i_1, \ldots, i_r) b_{i_1} \otimes \cdots \otimes b_{i_r},$$

be its expansion in the monomial basis. The set of monomials $b_{i_1} \otimes \cdots \otimes b_{i_r}$ such that the coefficients $g(b; i_1, \ldots, i_r)$ are nonzero is called the *monomial support* of b.

In analogy with the work of Kashiwara and Date et al mentioned above, it may be conjectured that:

Conjecture 2.2 (Existence of (local) crystal basis)

The $\tilde{Q}$-module $L^\rho_\alpha/qL^\rho_\alpha$ has a unique basis B^ρ_α (up to scaling by constant multiples, and which can be made completely unique by appropriate normalization) such that:

1. *The monomial supports of its elements are disjoint,*
2. *L^ρ_α and $B^\rho_\alpha \cup \{0\}$ are invariant under Kashiwara's crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for H_q (which are well defined since $U_q(\mathcal{H})$ is a subalgebra of $U^H_q(\mathcal{G})$), and $(L^\rho_\alpha, B^\rho_\alpha)$ is a local crystal basis, in the sense of Kashiwara, of $W_{q,\alpha}$ as an H_q-module*

Furthermore, this crystal base does not depend on the embedding ρ (up to isomorphism).

This conjecture has been supported by experimental evidence; cf. Section 5.5.

Assuming this, we let $(L_\alpha, B_\alpha) = (L^\rho_\alpha, B^\rho_\alpha)$ for any ρ as above. It is called the *upper crystal base* of $W_{q,\alpha}$.

In standard setting, the embedding $\rho : V_{q,\pi} \to V^{\otimes r}$ in Section 2.1 can be chosen so that the support of each basis element in B^ρ_π consists of just one monomial. That is, so that each $b \in B^\rho_\pi$ is a monomial in $B(V)^{\otimes r}$. In the nonstandard setting, it is not always possible to choose an embedding $\rho : W_{q,\alpha} \to X^{\otimes r}$ so that the support of each basis element in B^ρ_α in Conjecture 2.1 consists of just one monomial; cf. Section 5.5 for a counterexample.

In view of the nonstandard q-analogue of the Peter-Weyl theorem (2), $S = \mathcal{O}(M^H_q(X))$ has a natural upper crystal base

$$(L(S), B(S)) = \bigoplus_\alpha (L^*_\alpha, B^*_\alpha) \otimes (L_\alpha, B_\alpha) \tag{6}$$

at $q = 0$ (with appropriate normalization). Let S_Q be the K_Q-forms (K_Q-subring) of S generated by the entries u^i_j of the generic nonstandard quantum matrix $\mathbf{u}$; recall (cf. [GCT7]) that S is the quotient of $\mathbb{C}\langle \mathbf{u} \rangle$ modulo appropriate quadratic relations over the entries u^i_j's of $\mathbf{u}$. We define an involution $-$ over S by a natural generalization of its definition in the standard setting. Specifically, S is a $\mathcal{O}(M^H_q(X))$-bi-comodule. Via the homomorphism from

$\mathcal{O}(M_q^H(X))$ to $\mathcal{O}(M_q(V))$, S is also $\mathcal{O}(M_q(V))$-bi-comodule; i.e., an H_q-bi-module. In the spirit of [Kas2], for any u and v in S with H_q-bi-weights (λ_r, λ_l) and (μ_r, μ_l), let $\overline{uv} = q^{(\lambda_r,\mu_r)-(\lambda_l,\mu_l)}\bar{v}\bar{u}$, where (,) denotes the usual inner product in the H_q-weight space. We let $\bar{u}^i_j = u^i_j$, and $\bar{q} = q^{-1}$. This defines $-$ on S completely.

Applying $-$ to $(L(S), B(S))$, we get an upper crystal base $(\bar{L}(S), \bar{B}(S))$ at $q = \infty$. In analogy with the standard setting, we can now ask:

Question 2.3 *Is the triple $(S_Q, L(S), \bar{L}(S))$ balanced? In other words, is the map $\psi : E = S_Q \cup L(S) \cup \bar{L}(S) \to L(S)/qL(S)$ of $\hat{Q}$-modules an isomorphism.*

If it were, we could have defined the canonical basis of S by a globalization procedure very much as in the standard setting, i.e., as $\psi^{-1}(B(S))$. But, as it turns out, this need not be so; cf. Section 5.3. Specifically, for a given $b \in B(S)$, the fibre $\psi^{-1}(b)$ need not be singleton. This is the major difference from the standard setting that makes construction of the canonical basis of S in the nonstandard setting much more complex. We turn to this in the next section.

2.3 Nonstandard globalization via minimization of degree complexity

We now give a conjectural procedure for choosing an unambiguously defined canonical element $y_b \in \psi^{-1}(b)$, $b \in B(S)$. The set $\{y_b \mid b \in B(S)\}$ will then be the *canonical basis* of S.

Fix r. Let S_r denote the degree r component of S. Let $\mathcal{A} = \mathcal{A}_r^H = \mathcal{A}_r^H(q)$ be the nonstandard q-Schur algebra [GCT7], which is the dual $S_r^* = \mathrm{Hom}(S_r, K)$ of S_r. A polynomial irreducible G_q^H-module $W_{q,\alpha}$ of degree r is an irreducible $\mathcal{A}$-module, and conversely every irreducible $\mathcal{A}$-module is of this form. Furthermore, the q-analogue of the Peter-Weyl theorem also holds for $\mathcal{A}$:

$$\mathcal{A} = \oplus_\alpha W_{q,\alpha}^* \otimes W_{q,\alpha}. \tag{7}$$

For reasons given later (cf. Remark 1 below), it will be more convenient to construct the canonical basis of $\mathcal{A}_r^H$ first. The canonical basis of S_r will then be defined to be its dual.

Let

$$\langle\ \rangle : \mathcal{A} \otimes S_r \to K$$

be the natural pairing. The lattice $L(\mathcal{A})$ is defined to be the dual lattice of $L(S_r)$:

$$L(\mathcal{A}) = \{a \in \mathcal{A} \mid \langle a, L(S_r)\rangle \subseteq A_K\}.$$

The automorphism $-$ of $\mathcal{A}$ is defined by:

$$\langle \bar{a}, s\rangle = \langle a, \bar{s}\rangle^{-}.$$

We define $\bar{L}(\mathcal{A})$ by applying $-$ to $L(\mathcal{A})$. We define the Q-form (i.e. K_Q-form) $\mathcal{A}_Q$ of $\mathcal{A}$ by:

$$\mathcal{A}_Q = \{a \in \mathcal{A} \mid \langle a, S_{r,Q}\rangle \subseteq K_Q\},$$

where $S_{r,Q} = S_r \cap S_Q$. We define the basis $B(\mathcal{A})$ of $L(\mathcal{A})/qL(\mathcal{A})$ to be the dual of $B(S)$. Thus $(L(\mathcal{A}), B(\mathcal{A}))$ is the local crystal basis of $\mathcal{A}$ as per Conjecture 2.2 and we have an analogue of (6):

$$(L(\mathcal{A}), B(\mathcal{A})) = \bigoplus_{\alpha} (L_\alpha^*, B_\alpha^*) \otimes (L_\alpha, B_\alpha) \tag{8}$$

This defines the triple $(\mathcal{A}_Q, L(\mathcal{A}), \bar{L}(\mathcal{A}))$ for $\mathcal{A}$. It need not be balanced in the standard sense, just like the triple $(S_Q, L(S), \bar{L}(S))$ above. Now we describe the conjectural construction of a canonical basis of $\mathcal{A}$.

Each component $B_\alpha^* \otimes B_\alpha$ of $B(\mathcal{A})$ has the left and right action of Kashiwara's crystal operators for H_q (cf. Conjecture 2.2). Thus, by [Kas1], we get a crystal graph on $B_\alpha^* \otimes B_\alpha$, whose each connected component intutively corresponds to an irreducible H_q-bi-submodule $V_{q,\mu_1}^* \otimes V_{q,\mu_2}$ of the component $W_{q,\alpha}^* \otimes W_{q,\alpha}$ in the Peter-Weyl decomposition of $\mathcal{A}$ (7). With each element $b \in B(\mathcal{A})$ that occurs in such a connected component, we associate the triple $T(b) = (\alpha, \mu_1, \mu_2)$. We call it the *type* of b. The types $T(b)$'s can be partially ordered as follows.

First, put a partial order $\leq$ on the labels α of polynomial irreducible $\mathcal{A}$-modules $W_{q,\alpha}$ as follows. Consider $W_{q,\alpha}$ as an H_q-module. Let $\mu(\alpha)$ denote the highest weight in $W_{q,\alpha}$ as an H_q-module. There may be several highest weight vectors in $W_{q,\alpha}$, since $W_{q,\alpha}$ need not be irreducible as an H_q-module. That is fine. Let $\leq$ denote the usual partial order on the highest weights of H_q-modules: $\mu_1 \leq \mu_2$ iff $\mu_2 - \mu_1 \in \sum_\tau \mathbb{N}\tau$ where τ ranges over the simple positive roots of H. We say $\alpha \leq \alpha'$ iff $\mu(\alpha) \leq \mu(\alpha')$.

Now observe that each component $W^*_{q,\alpha} \otimes W_{q,\alpha}$ in the Peter-Weyl decomposition of $\mathcal{A}$ is an H_q-bimodule; i.e., has a left and right action of H_q. With each irreducible H_q-bimodule in this component isomorphic to $V_{q,\mu_1} \otimes V_{q,\mu_2}$, where μ_1 and μ_2 are highest left and right weights of H_q, we associate the type $T = (\alpha, \mu_1, \mu_2)$. Put a partial order, which we shall again denote by $\leq$, on the types T as per the partial order $\leq$ on the individual components. The type $T(b)$ associated with each $b \in B(\mathcal{A})$ above is similar to this type. So this also puts a partial order on the types $T(b)$'s.

Next fix a $b \in B(\mathcal{A})$. Let $T = T(b)$ be its type. We shall associate a canonical basis element y_b with each such b by induction on its type using the preceding partial order. The set $\{y_b\}$ will then be the required canonical basis of $\mathcal{A}$.

Let $\mathcal{A}^{\leq T}$ denote the span of all H_q-bimodules in $\mathcal{A}$ of types less than or equal to T as per $\leq$. Let

$$L(\mathcal{A}^{\leq T}) = L(\mathcal{A}) \cap \mathcal{A}^{\leq T}.$$

We define $\bar{L}(\mathcal{A})^{\leq T}$, and $\mathcal{A}_Q^{\leq T}$ similarly. Consider the natural projection

$$\psi_T : \mathcal{A}_Q^{\leq T} \cap L(\mathcal{A})^{\leq T} \cap \bar{L}(\mathcal{A})^{\leq T} \to L(\mathcal{A})^{\leq T}/qL(\mathcal{A})^{\leq T}.$$

Let $\psi_T^{-1}(b)$ be the fibre of b. If this fibre were to contain a unique element, then we can simply let y_b be this unique element. But this need not be so, because the triple $(\mathcal{A}_Q^{\leq T}, L(\mathcal{A})^{\leq T}, \bar{L}(\mathcal{A})^{\leq T})$ need not be balanced. So we have to resolve the ambiguity in some canonical way. Towards this end, we shall associate with each element in $\mathcal{A}_Q \cap L(\mathcal{A})$ a complexity measure, called its *degree complexity*. We shall then define y_b to be the element in $\psi_T^{-1}(b)$ of minimum degree complexity–it would be conjecturally unique; cf. Conjecture 2.5 below. This scheme is in the spirit of [KL1] where each element of the Kazhdan-Lusztig basis of the Hecke algebra is defined to be an element of minimum degree in a certain sense.

So let us define the degree complexity of an element $y \in \mathcal{A}_Q \cap L(\mathcal{A})$. Since $X_q^{\otimes r}$ is a represention of $\mathcal{A} = \mathcal{A}_r^H(q)$, we have the injection

$$\eta : \mathcal{A} \hookrightarrow Z = \mathrm{End}(X_q^{\otimes r}) = (X_q^{\otimes r})^* \otimes X_q^{\otimes r}.$$

Let $L(Z) = L(X_q^{\otimes r})^* \otimes L(X_q^{\otimes r})$ be the lattice associated with Z. The Q-form (or rather K_Q-form) Z_Q is defined similarly. Then

Proposition 2.4 *The embedding η injects the Q-form $\mathcal{A}_Q$ into Z_Q. Furthermore, assuming Conjecture 2.2, η also injects the lattice $L(\mathcal{A})$ into $L(Z)$.*

The proof is easy. (To be filled in).

Fix the upper canonical basis $\{x_i\}$ of X_q as an H_q-module. Let $\{x_i^*\}$ be the dual canonical basis of X_q^*. This fixes the upper canonical basis $CB(Z)$ of Z, whose each element is of the form

$$z_{i_1,\ldots,i_r;j_1,\ldots,j_r} = x_{i_1}^* \otimes \cdots \otimes x_{i_r}^* \otimes x_{j_1} \otimes \cdots \otimes x_{j_r}.$$

It is also a basis of the Q-form Z_Q and the lattice $L(Z)$. Now given any $y \in \mathcal{A}$, let $w = \eta(y)$. Express w in the canonical basis of Z:

$$w = \eta(y) = \sum_z a(y,z)z, \tag{9}$$

where z ranges of the basis elements in $CB(Z)$. Since $w \in L(Z) \cap Z_Q$, each $a(y,z) \in A_K \cap K_Q$. This means it is integral over $Q[q]$ and hence has a well defined degree $d(y,z)$ at $q = 0$ (the same as the order of its pole at $q = \infty$); if $a(y,z) = 0$, we define $d(y,z) = -\infty$. We define *the degree complexity* $d(y)$ of y to be the tuple $\langle \ldots, d(y,z), \ldots \rangle$ of these degrees. We put a partial order on the degree complexity as follows.

Let $U = U_q(H)$ be the Drinfeld-Jimbo enveloping algebra of H_q, U^- the subalgebra generated by its generators F_i's. For any string $\nu = \nu_1, \nu_2, \ldots$ of positive integers, let U_ν^- be the subspace of U^- spanned by the words in F_i's in which each F_i occurs occurs ν_i times. Given a canonical basis element x of X_q, we define its length to be $|x| = \sum_i \nu_i$, where $x \in U_\nu^- x_0$, and x_0 is the highest weight vector of X_q. Order the canonical basis elements of X_q as per the reverse order on their lengths; so that x_0 is the highest element in this order. Put a similar order on X_q^*. This puts an induced partial order on the elements of the canonical basis $CB(Z)$ of Z. We let $<$ denote the strict less than relation as per this partial order. Given y and y', and letting $w = \eta(y), w' = \eta(y')$, we say that $d(y) \leq d(y')$ if for every z: either $d(y,z) \leq d(y',z)$, or for some $\bar{z} < z$ $d(y,\bar{z}) < d(y',\bar{z})$.

Conjecture 2.5 (Minimum degree) *The fibre $\psi_T^{-1}(b)$ contains a unique element y_b of minimum degree complexity. Minimum means $d(y_b) \leq d(y)$, as per the ordering on the degree tuples above, for any $y \in \psi_T^{-1}(b)$.*

We call y_b the *canonical basis element* associated with b, and the set $\{y_b\}$ the *canonical basis* $CB(\mathcal{A})$ of $\mathcal{A}$. The canonical basis $CB(S_r) = \{x_b\}$ of S_r is defined to be its dual. The canonical basis $CB(S)$ of S is $\cup_r CB(S_r)$.

Remark 1: The reader may wonder why we defined the canonical basis of $\mathcal{A}$ first, and that of S_r later, as its dual. Can we define the canonical basis of S_r

directly? The algorithm for S_r would be similar as above. The main problem is to define the degree complexity of an element $x \in L(S_r) \cap S_Q$. We have a natural projection from $Z = \mathrm{End}(X_q^{\otimes d})$ to S_r, but not a natural injection that injects $S_{r,Q} = S_r \cap S_Q$ into Z_Q. So the analogue of Proposition 2.4 does not hold.

Remark 2: In the standard settting, Kashiwara and Lusztig give an efficient scheme for constructing each canonical basis element of the standard matrix coordinate ring $\mathcal{O}(M_q(V))$. We do not have here an analogous efficient algorithm for constructing the canonical basis element y_b in Conjecture 2.5. In the standard setting an efficient cosntruction was possible because the standard Drinfeld-Jimbo universal enveloping algebra has an explicit presentation in terms of generators and defining relations. This explicit presentation is crucially used in the construction of the standard canonical basis and also in proving correctness of the construction.

The nonstandard universal algebra does not have an analogous explicit presentation as yet; cf. [GCT7]. For this we need explicit formulae for the coefficients of the Laplace relation in [GCT7] among the simplest nonstandard canonical basis elements in S, namely nonstandard minors, since as discussed in [GCT7], it is the mother relation in the representation theory of the nonstandard quantum group (just as in the standard setting). That is, we need explicit interpretation for these coefficients in the spirit of the explicit interpretation for the coefficients of the Kazhdan-Lusztig polynomials in terms perverse sheaves. This is the basic core problem that needs to be solved to prove that the preceding algorithm for constructing the nonstandard canonical basis is correct and to give an explicit, efficient construction of y_b. Furthermore, if explicit presentation of the nonstandard universal algebra is so nonelementary, as against the elementary explicit presentation of the standard (Drinfled-Jimbo) enveloping algebra, then the task of proving correctness may be formidable.

2.4 Properties of the nonstandard canonical basis

It may be conjectured that the nonstandard canonical bases $CB(S)$ and $CB(\mathcal{A})$ have properties akin to the standard canonical basis of $\mathcal{O}(M_q(V))$:

2.4.1 Cellular decomposition

Conjecture 2.6 (Cell decomposition) *The refined Peter-Weyl theorem, akin to the one proved by Lusztig [Lu2] in the standard setting, holds for*

$CB(S)$ and $CB(\mathcal{A})$.

This means the left, right and two-sided cells in $\mathcal{O}(M_q^H(X))$ with respect to $CB(S)$ yield irreducible left, right, and two-sided (polynomial) representations of G_q^H. And furthermore, the left sub-cells of each left cell with respect to the restricted H_q-action yield irreducible H_q-representations. The left cell of $\mathcal{O}(M_q^H(X))$ is defined as follows. Given $b \in CB(S)$, let $\Delta(b) = \sum_{b',b''} c_b^{b',b''} b' \otimes b''$, where Δ denotes comultiplication. Then we say that $b'' \leftarrow_L b$ if $c_b^{b',b''}$ is nonzero for some b'. Let $<_L$ denote the transitive closure of $\leftarrow_L$. Using $<_L$ we define left cells in a natural way. The right and two-sided cells are defined similarly. The left, right and two-sided sub-cells with respect to the H_q-action are defined similarly. The definitions for $CB(\mathcal{A})$ are similar.

By restricting the canonical basis to any left cell corresponding to an irreducible polynomial representation $W_{q,\alpha}$ of G_q^H, we get the canonical basis of $W_{q,\alpha}$; here the choice of the left cell would conjecturally not matter up to scaling.

2.4.2 Positivity in the Kronecker case

Conjecture 2.7 (Positivity) *In the Kronecker case–i.e. when $H = GL(V) \times GL(W)$, $X = V \otimes W$ with the natural H-action–each coefficient $g(q)$ of any canonical basis element in $CB(\mathcal{A}_r^H)$ in the basis $CB(Z)$ (cf. eq.(9)) is a positive polynomial in q.*

Similarly, each multiplicative or comultiplicative structural constant of $CB(\mathcal{A}_r^H)$ is of the form $\stackrel{+}{-} (q - \frac{1}{q})^a f(q)$, where a is a nonnegative integer and $f(q)$ is a $-$-invariant positive and unimodal polynomial in q and q^{-1}.

The same also for $CB(S)$.

Here by a positive polynomial we mean a polynomial with nonnegative rational coefficients. By unimodality of the ($-$-invariant) polynomial $f(q)$, we mean its coefficients $f_{-k}, \ldots, f_k$ satisfy the condition

$$f_{-k} \leq f_{-k+1} \leq \cdots \leq f_{-1} \leq f_0 \geq f_1 \geq \cdots f_k.$$

By multiplicative structural constants, we mean the coefficients $m_{b,b'}^{b''}$ in the expansion

$$bb' = \sum_{b''} m_{b,b'}^{b''} b'',$$

for $b, b' \in CB(S)$, and with b'' ranging over the elements in $CB(S)$. Comultiplicative structural constants are defined similarly.

For experimental evidence for the dual nonstandard algebra $\mathcal{B}_r^H(q)$, see Section 5.1.

Presumably, the nonnegative coefficients of $g(q)$ and $f(q)$ may have a topological interpretation in the spirit of that for the coefficients of the Kazhdan-Lusztig polynomials, and unimodality of $f(q)$ may be a consequence of some result akin to the Hard Lefschetz theorem in the spirit of the results on unimodal sequences in [St].

The Kronecker case is fundamental because $\mathcal{O}(M_q^H(X))$ therein is a nonflat deformation of $\mathcal{O}(M(X))$, though $\mathbb{C}_q^H[X]$ is a flat deformation of $\mathbb{C}[X]$. Thus to prove the positivity Conjecture 2.7, some nonstandard extension of the theory of perverse sheaves and the work surrounding the Riemann hypothesis over finite fields [BBD, Dl2] that can deal with nonflat, noncommutative varieties like $\mathcal{O}(M_q^H(X))$ may be neeeded.

2.4.3 Nonstandard positivity and saturation

In the Kronecker case, the braided symmetric algebra $\mathbb{C}_q^H[X]$ is a flat deformation of $\mathbb{C}[X]$; i.e. $\dim(C_q^{H,r}[X]) = \dim(\mathbb{C}^r[X])$. But when $\mathbb{C}_q^H[X]$ is a nonflat deformation the situation is much more complex. To see this, let c be a coefficient of any canonical basis element in $CB(\mathcal{A}_r^H)$ in the basis $CB(Z)$. As observed after eq.(9), it is integral over $Q[q]$. Hence every coefficient of its minimal polynomial f_c is a polynomial in q with rational coefficients. In the spirit of Conjecture 2.7, one may ask if the coefficients of this polynomial are always nonnegative. Unfortunately, this need not be so; cf. Section 5.3 for counterexamples in the dual setting of $\mathcal{B}_r^H(q)$. The following is a relaxed version of Conjecture 2.7 for the general case.

Conjecture 2.8 (Saturation) *Each coefficient $a(q)$ of f_c is a* saturated *polynomial (in the terminology of [GCT6]); this means $a(1)$ is a positive rational if $a(q)$ is not an identically zero polynomial.*

Similarly, each coefficient $b(q)$ of the minimal polynomial of a multipliciative or comultiplicative structural constant of $CB(\mathcal{A}_r^H)$ can be expressed in the form

$$b(q) = (-1)^e (q - \frac{1}{q})^{e'} c(q) \tag{10}$$

for some nonnegative integers e, e', where

1. *e is chosen so that the middle term of $c(q)$ is positive; here by the middle term we mean the coefficient of q^i for the smallest i such that this coefficient is nonzero, and*

2. *$c(q)$ is a saturated polynomial in q and q^{-1}–this again means $c(1)$ is a positive rational if $c(q)$ is not an identically zero polynomial.*

In the context of the plethysm problem, one is finally interested in the behaviour of $b(q)$ at $q = 1$ (cf. Section 6), so this relaxed saturation form of positivity should be sufficient; see also [GCT6] for the importance of saturation in the context of the flip in GCT. A stronger positivity conjecture that would specialize to Conjecture 2.7 in the Kronecker case is:

Conjecture 2.9 (Nonstandard Positivity) *Each polynomial $c(q)$ in Conjecture 2.8 is almost positive and unimodal. That is, it is of the form $c^0(q) + c^1(q)$, where, if $c(q)$ is not identically zero,*

1. *$c^0(1) >> |c^1(1)|$, where $>>$ means much greater as $r \to \infty$, and more generally,*

2. *$c^0(q)$ is a dominant positive unimodal polynomial, and $c^1(q)$ is a very small error-correction polynomial. Specifically, $||c^1(q)||/||c^0(q)|| \leq 1/poly(\langle\mu\rangle, \langle\pi\rangle, \langle r\rangle)$, where μ and π are as in the plethysm problem (cf. begining of Section 1), $\langle\ \ \rangle$ denotes the bitlength-of-specification function, and $poly(\ \ )$ means polynomial of a fixed (constant) degree in the specified bitlengths, and $||\ \ ||$ denotes the L_2-norm of the coefficient vector of the polynomial.*

See Section 5.3 for experimental evidence for Conjectures 2.8-2.9 in the dual setting of $\mathcal{B}_r^H(q)$.

Presumably, the nonnegative coefficients of such $c^0(q)$ may again have a topological interpretation in the spirit of that for the coefficients of the Kazhdan-Lusztig polynomials, and unimodality of $c^0(q)$ may again be a consequence of some result akin to the Hard Lefschetz theorem. The correction polynomial $c^1(q)$ may also have a topological interpretation that depends on a cohomological measure of nonflatness of $C_q^H[X]$. In the Kronecker case, when $\mathbb{C}_q^H[X]$ is a flat deformation of $\mathbb{C}[X]$, this correction would then vanish, and Conjecture 2.9 would reduce to Conjecture 2.7. Furthermore, the conjectural nonnegative value of $c(1)$ may also have an interpretation akin to the representation-theoretic interpretation for the values of the Kazhdan-Lusztig polynomials at $q = 1$.

If nonstandard extension of the work surrounding the Riemann hypothesis over finite fields as needed to prove the positivity Conjecture refcposkroneckerleft in the Kronecker case can be found, that may open the way for investigating the more complicated nonstandard form of positivity in Conjecture 2.9.

3 Nonstandard canonical basis of $\mathcal{B}_r^H$

Let $\mathcal{B}_r^H = \mathcal{B}_r^H(q)$ be the nonstandard quantization of the symmetric group ring $\mathbb{C}[S_r]$ in [GCT7]. In this section, we describe an analogous conjecturally correct algorithm for constructing a nonstandard canonical basis $E(r)$ of $\mathcal{B}_r^H$. In the standard setting–i.e., when $H = G$–this basis would conjecturally specialize to the Kazhdan-Lusztig basis of the Hecke-algebra, though the specialized algorithm here is different from the algorithm in [KL1].

Since $\mathcal{B}_r^H(q)$ is semisimple [GCT4, GCT7], by the Wedderburn structure theorem

$$\mathcal{B}_r^H(q) = \bigoplus_\alpha T_{q,\alpha}^* \otimes T_{q,\alpha}, \tag{11}$$

where $T_{q,\alpha}$ ranges over the irreducible representations of $\mathcal{B}_r^H$, assuming that the underlying base field is a suitable algebraic extension of $Q(q^{1/2})$. We shall denote this base field by K–it is the same as the base field K in Section 2.2, except that the role of q there is played by $q^{1/2}$ here. Let $A_K, \bar{A}_K, K_Q, \hat{Q}$ be as in Section 2.2, with the role of q played by $q^{1/2}$.

We assume that H is the general linear group $GL(V)$ or a product of general linear groups. In this case (cf. [GCT7]), $\mathcal{B}_r^H$ is a $-$-invariant subalgebra of a suitable Hecke-algebra or a product of Hecke algebras, where $-$ denotes the usual bar-automorphism on the Hecke algebra [KL1] (analogue of $-$ in Section 2). Let $\mathcal{P}_i$'s and $\mathcal{Q}_i$'s denote the rescaled positive and negative generators of $\mathcal{B}_r^H$ as defined in [GCT7] (denoted by $p_{X,i}^{+,H}$ and $q_{X,i}^{+,H}$ therein) so that they belong to the usual $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-form on the ambient Hecke algebra (or the ambient product of Hecke algebras). Let $\mathcal{B}_{r,Q}^H$ denote the K_Q-form of $\mathcal{B}_r^H$ generated ring-theoretically by $\mathcal{P}_i$'s, or equivalently $\mathcal{Q}_i$'s.

The goal is to construct an A_K-lattice $R(r) \subseteq \mathcal{B}_r^H$ so that the canonical basis $E(r)$ of $\mathcal{B}_r^H$ can then be constructed by a nonstandard globalization procedure on the triple $(\mathcal{B}_{r,Q}^H, R(r), \bar{R}(r))$ analogous to the one Section 2.3. Just as in Section 2.2, it will turn out that this triple need not be balanced. We will resolve the ambiguity caused by lack of balance using the notion of minimum degree complexity very much as in Section 2.3.

3.1 Nonstandard Gelfand-Tsetlin basis of $\mathcal{B}_r^H(q)$

In the construction of Kazhdan-Lusztig polynomials as described in [KL1, So], the lattice in the Hecke algebra is constructed using its standard monomial basis. But in general $\mathcal{B}_r^H(q)$ does not have a naturally defined monomial basis; see [GCT4] for an example. So we need a different way to construct the lattice $R(r)$. The construction here will be analogous to the construction of the lattice in the standard matrix coordinate ring $\mathcal{O}(M_q(V))$ based on its Gelfand-Tsetlin basis. This construction in $\mathcal{O}(M_q(V))$ is different from the one defined in Section 2.1. We shall recall it using the same notation as in Section 2.1. It is based on the observation that the Gelfand-Tsetlin basis of an irreducible H_q-representation $V_{q,\mu}$ (after rescaling as described in section 7.3.3 of [Kli]) is its local crystal basis: i.e., it is an A-basis of the lattice $L_\mu \subseteq V_{q,\mu}$, and that its projection in L_μ/qL_μ is equal to the basis B_μ (whose elements have disjoint monomial supports as described before). This observation was in fact the starting point for the theory of local crystal basis [Kas1]. Let us denote the (rescaled) Gelfand-Tsetlin basis of $V_{q,\mu}$ by $GT_{q,\mu}$. The Gelfand-Tsetlin basis of $\mathcal{O}(M_q(V))$ is defined as per the standard Peter-Weyl theorem (3):

$$GT(\mathcal{O}(M_q(V))) = \bigcup_\mu GT^*_{q,\mu} \otimes GT_{q,\mu}. \tag{12}$$

Let L_{GT} be the lattice generated by this Gelfand-Tsetlin basis, and B_{GT} the projection of the Gelfand-Tsetlin basis on L_{GT}/qL_{GT}. Then (L_{GT}, B_{GT}) coincides with the standard crystal base (L, B) of $\mathcal{O}(M_q(V))$ in Section 2.1.

The algebra $\mathcal{B}_r^H(q)$ has a natural analgoue of the Gelfand-Tseltin basis, which can then be used to construct the lattice $R(r)$. We begin by describing this basis for an irreducible representation $T_{q,\alpha}$ of $\mathcal{B}_r^H(q)$.

We proceed by induction on r, the case $r = 1$ being easy. The following is a conjectural analogue of the standard Pieri's rule in this setting:

C1': $T_{q,\alpha}$ has a multiplicity-free decomposition as a $\mathcal{B}_{r-1}^H$-module.

(This conjecture is not really necessary as long as there is a natural way to resolve the ambiguity caused by multiplicity). By induction, we have defined a basis for each irreducible $\mathcal{B}_{r-1}^H(q)$-submodule of $T_{q,\alpha}$. Putting these bases together, we get the sought *nonstandard Gelfand-Tsetlin basis* C_α of $T_{q,\alpha}$.

Assuming multiplicity-free decomposition, such a basis is unique, up to scaling factors, which will be fixed in the course of the algorithm below.

Each element $x \in C_\alpha$ can be indexed by a *nonstandard Gelfand-Tsetlin tableau*, which is an analogue of the standard Gelfand-Tsetlin tableau [Kli] in this setting. It is defined to be the tuple $(\alpha_r, \alpha_{r-1}, \ldots)$, with $\alpha_r = \alpha$, of the classifying labels–which we shall call *types*–of the irreducible $\mathcal{B}_i^H$-submodules T_{q,α_i} containing x, where $T_{q,\alpha_i} \subset T_{q,\alpha_{i+1}}$.

We define the *nonstandard Gelfand-Tsetlin basis* $C(r)$ of $\mathcal{B}_q^H(r)$ as per the decomposition (11):

$$C(r) = \bigcup_\alpha C_\alpha^* \otimes C_\alpha.$$

Each element of $C(r)$ is indexed by a *nonstandard Gelfand-Tsetlin bi-tableau* as per this decomposition. We shall denote the element of $\mathbb{C}(r)$ indexed by a nonstandard Gelfand-Tsetlin bitableau T by c_T.

3.2 Local crystal base

The sought lattice $R(r) \subseteq \mathcal{B}_r^H(q)$ will be generated by the elements of $C(r)$ after scaling them appropriately in the course of the algorithm below. Let us assume at the moment that this scaling has already been given to us, and thus $R(r)$ is fixed. Let b_T denote the image of c_T under the projection $\psi : R(r) \to R(r)/q^{1/2}R(r)$. Let $B(r) = \{b_T\}$ be the basis of $R(r)/q^{1/2}R(r)$. Then $(R(r), B(r))$ is the analgoue of the local crystal base in the standard setting.

3.3 Nonstandard globalization via minimization of degree complexity

The elements of $C(r)$ need not be $-$-invariant. Next we globalize $C(r)$ to get a $-$-invariant canonical basis $E(r)$ of $\mathcal{B}_r^H$ in the spirit of Kazhdan and Lusztig [KL1], with the role of the standard basis in [KL1, So] played by the nonstandard Gelfand-Tsetlin basis of $\mathcal{B}_r^H(q)$ here. As already mentioned, the main difference from the standard setting of Hecke algebra is that $(\mathcal{B}_r^H(q), R(r), \bar{R}(r))$ need not be balanced. This is the main problem that needs to be addresssed. The nonstandard globalization procedure here is analogous to the one in Section 2.3. It goes as follows.

(1) In Section 2.3, we described a partial order $\leq$ on the types (classifying labels) of the irreducible modules $W_{q,\alpha}$ of $\mathcal{A}_r^H(q)$. By the nonstadard duality conjecture [GCT7], this induces a partial order $\leq$ on the types (classifying labels) of the (paired) irreducible modules $T_{q,\alpha}$ of $\mathcal{B}_r^H(q)$. (In the standard

setting, this procedure would yield a partial order on the partitions of size r, with the partition containing a single row of size r at the top of the order and the partition containing a single column of size r at the bottom of the order.)

(2) This induces a lexicographic partial order $\leq$ on the nonstandard Gelfand-Tsetlin tableaux, since they are just tuples of types, and also on nonstandard Gelfand-Tsetlin bitableau which index the basis elements $C(r)$.

(3) Let $\mathcal{B}^{\leq T}$ be the span of the basis elements $c_{T'} \in C(r)$ such that $T' \leq T$. Let $R^{\leq T} = R(r) \cap \mathcal{B}^{\leq T}$, $\bar{R}^{\leq T} = \bar{R}(r) \cap \mathcal{B}^{\leq T}$, and $\mathcal{B}_Q^{\leq T} = \mathcal{B}_{r,Q}^H \cap \mathcal{B}^{\leq T}$. Then the triple $(\mathcal{B}_Q^{\leq T}, R^{\leq T}, \bar{R}^{\leq T})$ need not be balanced. To define a canonical basis element e_T associated with T, we associate a *degree complexity* with each element $y \in \mathcal{B}_{r,Q}^H$ in the spirit of Section 2.3.

This is done as follows. Since we are assuming that H is $GL(V)$ or a product of general linear groups, $\mathcal{B}_r^H$ is a subalgebra of a product of Hecke algebras, say $Z = \mathcal{H}_{k_1}(q) \otimes \cdots \otimes \mathcal{H}_{k_l}(q)$, where $\mathcal{H}_j(q)$ denotes the Hecke algebra with rank j. Furthermore,

$$\mathcal{B}_{r,Q}^H \subseteq Z_Q = \mathcal{H}_{k_1,Q} \times \cdots \times \mathcal{H}_{k_l,Q},$$

where $\mathcal{H}_{j,Q}$ denote the K_Q-form of $\mathcal{H}_j(q)$ obtained by tensoring its usual $Q[q^{1/2}, q^{-1/2}]$-form with K_Q. Consider the Kazhdan-Lusztig basis $KL(Z)$ of Z formed by taking the product of the Kazhdan-Lusztig bases of its Hecke algebra factors. Express $y \in \mathcal{B}_{r,Q}^H$ in terms of $KL(Z)$:

$$y = \sum_z a(y,z) z, \tag{13}$$

where z ranges over the elements in $KL(Z)$. Then each coefficient $a(y,z) \in K_Q$. Let $d(y,z)$ denote the degree of $a(y,z)$; i.e., the order of its pole at $q = \infty$. If $a(y,z) = 0$, we define $d(y,z) = -\infty$. We define *the degree complexity* $d(y)$ of y to be the tuple $\langle \ldots, d(y,z), \ldots \rangle$ of these degrees. We put a partial order on degree complexities as follows. Put a partial order $\leq$ on the Kazhdan-Lusztig basis of the Hecke algebra $\mathcal{H}_j(q)$ as per the reverse order on the (reduced) lengths of the permutation indices of the basis elements–so 1 is the highest element as per this order. This also puts a partial order on $KL(Z)$. Let $<$ denote the strict less than relation as per this partial order. Given y and y', we say that $d(y) \leq d(y')$ if for every z: either $d(y,z) \leq d(y',z)$, or for some $\bar{z} < z$, $d(y,\bar{z}) < d(y',\bar{z})$.

Consider the natural projection

$$\psi_T : R^{\leq T} \cap \bar{R}^{\leq T} \cap \mathcal{B}_Q^{\leq T} \to R^{\leq T}/q^{1/2} R^{\leq T}.$$

Let $\psi_T^{-1}(b_T)$ be the fibre of $b_T \in B(r)$. The following is the analogue of Conjecture 2.5 in this context (with different interpretation for b, y, ψ etc. from there):

Conjecture 3.1 (Minimum degree) *The fibre $\psi_T^{-1}(b_T)$ contains a unique element e_T of minimum degree complexity. Minimum means $d(e_T) \leq d(y)$, as per the ordering on the degree tuples above, for any $y \in \psi_T^{-1}(b_T)$.*

We call e_T the canonical basis element associated with T, and $E(r) = \{e_T\}$ the canonical basis $\mathcal{B}_r^H(q)$.

So far we have not discussed how to scale the nonstandard Gelfand-Tsetlin basis of $\mathcal{B}_r^H(q)$ to get the lattice $R(r)$. To complete the algorithm, it remains to fix this scaling.

Let $\{c'_T\}$ denote the nonstandard Gelfand-Tsetlin basis of $\mathcal{B}_r^H(q)$ before scaling. The scaled c_T will be of the form $q^{a_T} c'_T$ for some rational a_T. We have to determine all a_T's. Assume that $a_{T'}$, $T' < T$, have been fixed. For any rational a, let $c_{a,T} = q^a c'_T$. Let $R^{a,\leq T}$ be the lattice generated by $c_{a,T}$ and $c_{T'}$, $T < T$, and $\bar{R}^{a,\leq T}$ obtained by applying $-$ to it. Consider the projection

$$\psi_{a,T} : R^{a,\leq T} \cap \bar{R}^{a,\leq T} \cap \mathcal{B}_{\mathcal{Q}}^{\leq T} \to R^{a,\leq T}/q^{1/2} R^{a,\leq T}.$$

Let $b_{a,T}$ be the image of $c_{a,T}$ under the projection $R^{a,\leq T}/q^{1/2} R^{a,\leq T}$. Let $\psi_{a,T}^{-1}(b_{a,T})$ be its fibre. The following is the strengthened form of Conjecture 2.5.

Conjecture 3.2 (Minimum degree) *There exists a unique a_T and $e_T \in \psi_{a_T,T}^{-1}(b_{a_T,T})$ such that for any a and any $y \in \psi_{a,T}^{-1}(b_{a,T})$ $d(e_T) \leq d(y)$. That is, e_T is the unique element of minimum degree complexity over all choices of a.*

This fixes a_T. Furthermore, $\psi_T = \psi_{a_T,T}$, $b_T = b_{a_T,T}$, and e_T in Conjecture 3.1 is the same as here.

If instead of the order $\leq$ among the classifying labels α's of $T_{q,\alpha}$, we use its reverse order, and in the definition of degree complexity use the opposite of the Kazhdan-Lusztig basis (obtained by replacing $\mathcal{Q}_i$ by $\mathcal{P}_i$), we get another canonical basis $E^{opp}(r)$ of $\mathcal{B}_r^H(q)$, which we shall call its *opposite canonical basis.*

To prove Conjectures 3.1-3.2, we need to know relations among the generators $\mathcal{P}_i$'s of $\mathcal{B}_r^H$ explicitly, just as we know the relations among the generators of the Hecke algebra explicitly. This is not known at present. See [GCT7] for the problems that arise in this context.

Each element c of the Kazhdan-Lusztig basis of the Hecke algebra can be expressed in the form

$$c = c_0 + \sum_{j>0} a(j) c_j,$$

where each c_j is a monomial in the generators of the Hecke algebra, $a(j) \in Q[q^{1/2}, q^{-1/2}]$ and the length of each c_j, $j > 0$, is smaller than that of c_0. This need not be so in the nonstandard setting: there can be several monomials of maximum length with nontrivial coefficients in any monomial representation of a nonstandard canonical basis element; cf. Section 5.3 and Figure 11 therein for an example.

3.4 Conjectural properties

It may be conjectured that the canonical bases $E(r)$ and $E^{opp}(r)$ have properties akin to those of the Kazhdan-Lusztig basis of the Hecke algebra.

3.4.1 Cellular decomposition

Conjecture 3.3 (Cell decomposition) *Analogue of the cell decomposition property of the Kazhdan-Lusztig basis also holds for $E(r)$ and $E^{opp}(r)$.*

Specifically this means the following. Let us define the left, right and two-sided cells of $\mathcal{B}_r^H$ with respect to the canonical basis $E(r)$ very much as in Section 2.4. Then it may be conjectured that they yield irreducible left, right and two-sided representations of $\mathcal{B}_r^H$. The conjecture for $E^{opp}(r)$ is similar.

By restricting $E(r)$ to any left cell corresponding to an irreducible $\mathcal{B}_r^H$-module $W_{q,\alpha}$, we get the canonical basis of $W_{q,\alpha}$; here the choice of the left cell would conjecturally not matter (up to scaling).

3.5 Positivity in the Kronecker case

The following is an analogue of Conjecture 2.7 here.

Conjecture 3.4 (Positivity) *Let $c \in K_Q$ be a multiplicative or comultiplicative structural constant of $E(r)$ or a structural coefficient of a canonical basis element in $E(r)$–i.e., a coefficient of its expression in terms of the canonical basis of Z as in eq.(13).*

In the Kronecker case, c is of the form $\overset{+}{-}(q^{1/2} - q^{-1/2})^a f(q)$, where a is a nonnegative integer and $f(q)$ is a $-$-invariant positive and unimodal polynomial in $q^{1/2}$ and $q^{-1/2}$.

The same for $E^{opp}(r)$.

For experimental evidence see Sections 5.1.2 and 5.2.

3.5.1 Nonstandard positivity and saturation

The general case is much more complex as in Section 2.4.3. The following is an analogoue of Conjecture 2.8.

Conjecture 3.5 (Saturation) *Let $c \in K_Q$ be a multiplicative or comultiplicative structural constant of $E(r)$ or a structural coefficient of a canonical basis element in $E(r)$–i.e., a coefficient of its expression in terms of the canonical basis of Z as in eq.(13). Let f_c be its minimal polynomial with coefficients in $Q(q^{1/2})$.*

Then each coefficient $s(q)$ of f_c can be expressed in the form

$$s(q) = (-1)^e (q^{1/2} - q^{-1/2})^{e'} g(q) \tag{14}$$

for some nonnegative integers e, e', where

1. *e is chosen so that the middle term of $g(q)$ is positive; and*
2. *$g(q)$ is a saturated polynomial in $q^{1/2}$ and $q^{-1/2}$.*

Analogue of the stronger Conjecture 2.9 in this case is:

Conjecture 3.6 (Nonstandard Positivity, informal) *Each polynomial $g(q)$ above is almost positive and unimodal; i.e. of the form $g^0(q) + g^1(q)$, where, if $g(q)$ is not identically zero,*

1. *$g^0(1) >> |g^1(1)|$, and more generally,*

2. *$g^0(q)$ is a dominant positive unimodal polynomial, and $g^1(q)$ is a very small error-correction polynomial. Specifically, $||g^1(q)||/||g^0(q)|| \leq 1/poly(\langle\mu\rangle, \langle\pi\rangle, \langle r\rangle)$, with the terminology as in Conjecture 2.9.*

For experimental evidence, see Section 5.4.1.

Here, $g(1)$ and the conjectural nonnegative coefficients of $g^0(q)$ may have a representation-theoretic/topological/cohomological interpretation akin to that sought for the analogous quantities in Section 2.4.3.

3.5.2 Quasi-cellular decomposition

Conjecture 3.7 *The opposite canonical basis E^{opp} also has the following quasi-cellular decomposition property.*

For this we define a quasi-subcellular decomposition of each left or right cell with respect to $E^{opp}(r)$. Specifically, given e', e'' belonging to the same left cell, express

$$ee' = \sum_{e''} \epsilon(e, e', e'')(q^{1/2} - q^{-1/2})^{\delta(e,e',e'')} d^{e''}_{e,e'} e'', \tag{15}$$

where $e'' \in E(r)$, the sign $\epsilon(e, e', e'')$ is either 1 or -1, $\delta(e, e', e'')$ is a nonnegative integer, and $d^{e''}_{e,e'}$ is a $-$-invariant saturated polynomial in $q^{1/2}$ and $q^{-1/2}$ as per the saturation Conjecture 3.5. We say that $e'' \propto_L e'$ if, for some e, $d^{e''}_{e,e'}$ in (15) is nonzero and $\delta(e, e', e'')$ is zero; i.e., if, for some e, e'' occurs with nonzero coefficient in the expansion of ee' specialized at $q = 1$. Let $\prec_L$ denote the transitive closure of $\propto_L$. Using $\prec_L$ we define left quasi-subcells of a left-cell of $E^{opp}(r)$. It may be conjectured that each left quasi-subcell of $E^{opp}(r)$ yields an irreducible representation of the symmetric group S_r at $q = 1$. That is, when the $\mathcal{B}^H_r$-representation Y corresponding this left cell is specialized at $q = 1$, so as to become a representation $Y_{q=1}$ of $\mathbb{C}[S_r]$, the partial order on its left quasi-sub-cells induces a composition series of $Y_{q=1}$ whose factors are irreducible representations of $\mathbb{C}[S_r]$.

Fix one such quasi-subcell C of $E^{opp}(r)$. Let $S_{\lambda(C)}$ be the irreducible representation (Specht module) of $\mathbb{C}[S_r]$ that is isomorphic to the factor in correspondence with C in this composition series of $Y_{q=1}$, where $\lambda(C)$ is a partition depending on C. The canonical basis elements in C, after specialization at $q = 1$ and projection, yield a basis of $S_{\lambda(C)}$. It may be conjectured that this basis coincides with the Kazhdan-Lusztig basis of $S_{\lambda(C)}$

(up to rescaling). By the Kazhdan-Lusztig basis of $S_{\lambda(C)}$, we mean specialization at $q = 1$ of the Kazhdan-Lusztig basis of the quantized Specht module $S_{q,\lambda(C)}$ of the Hecke algebra $\mathcal{H}_r(q)$.

But Conjecture 3.7 need not hold for $E(r)$; cf. Section 5.3 for a counterexample. This is analgous to the fact that the refined Peter-Weyl theorem in [Lu2] for the coordinate ring of the standard quantum group H_q holds only for the ordering $\leq$ (as defined in Section 2.3) among the labels (highest weights) of irreducible H_q-modules–there is no canonical basis of the standard coordinate ring which admits refined Peter-Weyl theorem for the opposite of the order $\leq$.

4 Internal definition of degree complexity

We give here an *internal* definition of degree complexity which may be used in place of the definition in Section 3.3 during the construction of the canonical basis. By internal, we mean it is based only on the structure of $\mathcal{B}_r^H(q)$ and does not depend on its embedding in the external ambient algebra Z there. This notion of degree complexity does not coincide with the one in Section 3.3, but the canonical basis constructed using this definition may be conjectured to be the same as the one constructed therein.

Let $\mathcal{B}[i] \subseteq \mathcal{B}_r^H(q)$ be the span of the monomials in $\mathcal{Q}_j$'s of length j, and $\mathcal{B}[< i]$ of length $< i$. We say that a given set of monomials in $\mathcal{Q}_j$'s of length i is *independent* if the images of these monomials in $\mathcal{B}[i]/\mathcal{B}[< i]$ are linearly independent. An expression

$$a = \sum_m a_m m, \tag{16}$$

where m ranges over monomials in $\mathcal{Q}_j$'s and $a_m \in K_Q$, is called *valid* if, for each i, the monomials m of length i with $a_m \neq 0$ in this expression are independent. Assume that a is $-$-invariant, so that each a_m is $-$-invariant. The degree complexity $\hat{d}(a)$ of a is defined to be the tuple $\langle \hat{d}_l(a), \ldots, \hat{d}_i(a), \ldots, \hat{d}_0(a) \rangle$ where $\hat{d}_i(a)$ denotes the maximum degree (at $q = 0$) of a_m for any m of length i, and l is the maximum length of m with $a_m \neq 0$ in the expression (16); by definition, $\hat{d}_i(a) = -\infty$ if there is no m in (16) of length i with $a_m \neq 0$. We order these degree complexities lexicographically. The degree complexity $\hat{d}(b)$ of an element $b \in \mathcal{B}_{r,Q}^H(q)$ is defined to be the minimum degree complexity of its any valid expression. It may be conjectured that if this definition of degree complexity, with the lexicographic ordering as above, is used in place of the definition of degree complexity in Section 3.3, the algorithm therein still works correctly and constructs the same canonical basis $E(r)$.

For the opposite canonical basis $E^{opp}(r)$, one can similarly use the internal definition as above with $\mathcal{P}_i$ in place of $\mathcal{Q}_i$.

The definition of degree complexity in this section is not as satisfactory as in Section 3.3 because $\mathcal{B}_q^H(r)$ does not a natural monomial basis [GCT4]. Hence to find the degree complexity of an element, one has to consider all its monomial expressions, finite but huge in number. It will be interesting to know if there is a more efficient internal definition.

5 Experimental evidence for $\mathcal{B}_r^H(q)$

In this section we shall verify the conjectures in this paper for two nontrivial special cases of the nonstandard algebra $\mathcal{B} = \mathcal{B}_r^H(q)$. The nonstandard canonical bases of $\mathcal{B}_r^H(q)$ in these cases were computed with the help of a computer using the algorithm in Section 3 and the notion of degree complexity as in Section 3.3.

First, some notation. Given a string $\sigma = i_1 \cdots i_k$ of positive integers, we let $\mathcal{P}_\sigma$ denote the monomial $\mathcal{P}_{i_1} \cdots \mathcal{P}_{i_k}$; $\mathcal{Q}_\sigma$ is defined similarly. Given a $-$-invariant polynomial $g(q) \in Q[q, q^{-1}]$, we define the *vector* a_g associated with $g(q)$ as follows. Express $g(q)$ in the form $\stackrel{+}{-} (q-1/q)^e h(q)$, where e is the maximum possible. Let $h_{-l}, \ldots, h_0, \ldots, h_l$ be the coefficients of $h(q)$. Then a_g is defined to be $[h_0, \ldots, h_l]$. In particular if $h(q)$ is (positive) unimodal, then a_q is a (positive) nonincreasing sequence. The vector associated with a $-$-invariant polynomial in $q^{1/2}$ and $q^{-1/2}$ is defined similarly.

5.1 Kronecker problem: $n = 2, r = 3$

Consider $\mathcal{B} = \mathcal{B}_3^H(q)$ in the special case of the Kronecker problem for $n = 2$ and $r = 3$. Thus $H = Gl_2 \times Gl_2$, and $G = Gl_4$ with H embedded diagonally. Let $\mathcal{P}_i$, $i = 1, 2$, be as in Section 3 and [GCT4]. The nonstandard canonical basis of $\mathcal{B}$ was computed in [GCT4] by an ad hoc method for $r = 3$, but it coincides with the one computed by the algorithm here. It is as follows. Let

$$c_1 = \frac{q^6 + 2q^5 + 3q^4 + 4q^3 + 3q^2 + 2q + 1}{q^3}, \qquad c_2 = \frac{q^4 + q^3 + 4q^2 + q + 1}{q^2},$$

$$b_1 = -(q^2 + 1)^2/q^2, \text{ and } b_2 = (q + 1)^2/q.$$

Then the opposite canonical basis $E^{opp}(3)$ of $\mathcal{B}$ consists of the following ten elements:

$$
\begin{array}{l}
\Sigma = c_1\mathcal{P}_1 - c_2\mathcal{P}_{121} + \mathcal{P}_{12121}, \\
\gamma_1^i = b_1\mathcal{P}_1 + \mathcal{P}_{121}, \quad i = 1,2, \\
\gamma_{12}^i = b_1\mathcal{P}_{12} + \mathcal{P}_{1212}, \quad i = 1,2, \\
\gamma_2^i = b_1\mathcal{P}_2 + \mathcal{P}_{212}, \quad i = 1,2, \\
\gamma_{21}^i = b_1\mathcal{P}_{21} + \mathcal{P}_{2121}, \quad i = 1,2, \\
\mu = 1.
\end{array} \tag{17}
$$

The canonical basis $E(3)$ is obtained by susbstituting $\mathcal{Q}_i$ for $\mathcal{P}_i$. In what follows, we shall only consider $E^{opp}(3)$.

5.1.1 Cellular and quasi-cellular decomposition

The basis $E^{opp}(3)$ has a cellular decomposition, in accordance with Conjecture 3.3, with the following right cells:

$$
\begin{array}{lcl}
U_\sigma & = & \{\Sigma\} \\
V_1 & = & \{\gamma_1^1, \gamma_{12}^1\} \\
V_2 & = & \{\gamma_1^2, \gamma_{12}^2\} \\
W_1 & = & \{\gamma_2^1, \gamma_{21}^1\} \\
W_2 & = & \{\gamma_2^2, \gamma_{21}^2\} \\
U_\mu = & = & \{\mu\}.
\end{array}
$$

The left cell decomposition is similar. The representation of $\mathcal{B}$ supported by U_σ is the trivial one dimensional representation. The representation supported by V_1 or W_1 is isomorphic; let us call it χ^1. Similarly, the representation supported by V_2 or W_2 is isomorphic; let us call it χ^2. Then χ^1 and χ^2 are two nonisomorphic two-dimensional representations of $\mathcal{B}$ which specialize at $q = 1$ to the two-dimensional Specht module of the symmetric group S_3 corresponding to the partition $(2,1)$. Thus quasi-cellular decomposition (Conjecture 3.7) holds trivially here.

5.1.2 Positivity

Coefficients of the elements of W_1 and W_2 in the Kazhdan-Lusztig basis of $\mathcal{H}_3(q) \otimes \mathcal{H}_3(q) \supseteq \mathcal{B}_3^H(q)$ are shown in Figure 1 (with the Kazhdan-Lusztig basis symmetrized and appropriately ordered as described in [GCT4]); the first column shows the coefficients of γ_1^1, the second of γ_{12}^1, and so on. It can be observed that all coefficients are positive, and unimodal polynomials in $Q[q, q^{-1}]$. The cofficients of other canonical basis elements can be found in [GCT4]; they too are positive, unimodal polynomials. This verifies the positivity Conjecture 3.4 for the structural coefficients of the canonical basis. A few typical nonzero multiplicative structural constants of the canonical basis are shown in Figure 2, where the coefficient of bb' with respect to the basis element b'' is denoted by $c(b, b'; b'')$. It can be seen that each constant is a polynomial of the form

$$(-1)^a (q^{1/2} - q^{-1/2})^b f(q^{1/2}, q^{-1/2}),$$

where f is a positive unimodal polynomial. It was verified with computer that all multiplicative structural constants are of this form. This verifies the positivity Conjecture 3.4 for the multiplicative structural constants as well.

5.2 Kronecker case, $H = SL_2$, $r = 4$

For the Kronecker case, $H = Gl_2 \times GL_2$, $G = GL_4$, and $r = 4$, we could compute just one canonical basis element Σ (akin to Σ in Section 5.1) corresponding to the trivial one dimensional representation of $\mathcal{B}_4^H(q)$. Symbolic computations needed to compute other canonical basis elements turned out to be beyond the scope of MATLAB/Maple on an ordinary workstation. The coefficients of Σ in the Kazhdan-Luztig basis of $\mathcal{H}_4(q) \otimes \mathcal{H}_4(q) \supset \mathcal{B}_4^H(q)$ were computed in MATLAB/Maple. There are 576 coefficients in total. Figures 3-5 show the vectors associated with distinct nonzero coefficients among these. They can be seen to be positive and nonincreasing in accordance with Conjecture 3.4.

5.3 $H = sl_2$, $G = sl_4$

Now we study the nonstandard algebra $\mathcal{B} = \mathcal{B}_3^H(q)$, when $H = Gl_2$, X is its four dimensional irreducible representation, and $G = GL(X) = Gl_4$. It is a 21-dimensional algebra whose explicit presentation is given in Section 7.1 of [GCT7]. We follow the notation as therein. Let $\mathcal{P}_i$ and $\mathcal{Q}_i$ be as defined in the begining of that section. The monomials $\mathcal{P}_\sigma$, where σ ranges over strings in 1 and 2 of length $k \leq 10$ with no consecutive 1's or 2's, form a basis of $\mathcal{B}$. This algebra has one trivial one-dimensinal representation, and five nonisomorphic two-dimensional representations, so that

$$21 = 1 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2.$$

Let

$$disc = \left(5\,q^{16} + 8\,q^{12} - 4\,q^{10} + 18\,q^8 - 4\,q^6 + 8\,q^4 + 5\right)\left(q^8 + 1\right)^2 q^{24}, \quad (18)$$

$$
\left[
\begin{array}{cccc}
\frac{\left(q^2+q+1\right)(1+q)^4}{q^3} & \frac{\left(q^2+q+1\right)(1+q)^6}{q^4} & \frac{\left(q^2+q+1\right)(1+q)^4}{q^3} & \frac{\left(q^2+q+1\right)(1+q)^6}{q^4} \\
\frac{(1+q)^5}{q^{5/2}} & \frac{(1+q)^7}{q^{7/2}} & \frac{(1+q)^5}{q^{5/2}} & \frac{(1+q)^7}{q^{7/2}} \\
\frac{(1+q)^4}{q^2} & \frac{(1+q)^6}{q^3} & \frac{(1+q)^4}{q^2} & \frac{(1+q)^6}{q^3} \\
\frac{(1+q)^3}{q^{3/2}} & \frac{(1+q)^5}{q^{5/2}} & \frac{(1+q)^3}{q^{3/2}} & \frac{(1+q)^5}{q^{5/2}} \\
\frac{(1+q)^5}{q^{5/2}} & \frac{(1+q)^7}{q^{7/2}} & \frac{(1+q)^5}{q^{5/2}} & \frac{(1+q)^7}{q^{7/2}} \\
\frac{(1+q)^4}{q^2} & \frac{(1+q)^6}{q^3} & \frac{(1+q)^4}{q^2} & \frac{(1+q)^6}{q^3} \\
2\,\frac{3\,q^3+1+3\,q+8\,q^2+q^4}{q^2} & 2\,\frac{\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)(1+q)^2}{q^3} & 2\,\frac{(1+q)^4}{q^2} & 2\,\frac{\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)(1+q)^2}{q^3} \\
\frac{(1+q)\left(q^2+6\,q+1\right)}{q^{3/2}} & 2\,\frac{(1+q)\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)}{q^{5/2}} & 2\,\frac{(1+q)^3}{q^{3/2}} & \frac{\left(q^2+6\,q+1\right)(1+q)^3}{q^{5/2}} \\
8 & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} & 2\,\frac{(1+q)^2}{q} & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} \\
\frac{\left(q^2+4\,q+1\right)(1+q)^2}{q^2} & \frac{\left(q^4+5\,q^3+12\,q^2+5\,q+1\right)(1+q)^2}{q^3} & \frac{\left(q^2+4\,q+1\right)(1+q)^2}{q^2} & \frac{\left(q^4+5\,q^3+12\,q^2+5\,q+1\right)(1+q)^2}{q^3} \\
\frac{(1+q)\left(q^2+6\,q+1\right)}{q^{3/2}} & \frac{\left(q^2+6\,q+1\right)(1+q)^3}{q^{5/2}} & 2\,\frac{(1+q)^3}{q^{3/2}} & 2\,\frac{(1+q)\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)}{q^{5/2}} \\
4\,\frac{(1+q)^2}{q} & 2\,\frac{3\,q^4+6\,q^3+14\,q^2+6\,q+3}{q^2} & 4\,\frac{(1+q)^2}{q} & 4\,\frac{(1+q)^4}{q^2} \\
4\,\frac{1+q}{\sqrt{q}} & 4\,\frac{(1+q)^3}{q^{3/2}} & 4\,\frac{1+q}{\sqrt{q}} & 4\,\frac{(1+q)^3}{q^{3/2}} \\
2\,\frac{(1+q)^3}{q^{3/2}} & 2\,\frac{(1+q)\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)}{q^{5/2}} & \frac{(1+q)\left(q^2+6\,q+1\right)}{q^{3/2}} & \frac{\left(q^2+6\,q+1\right)(1+q)^3}{q^{5/2}} \\
2\,\frac{(1+q)^2}{q} & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} & 2\,\frac{(1+q)^2}{q} & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} \\
8 & 8\,\frac{(1+q)^2}{q} & 8 & 8\,\frac{(1+q)^2}{q} \\
2\,\frac{(1+q)^2}{q} & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} & 8 & \frac{\left(q^2+6\,q+1\right)(1+q)^2}{q^2} \\
4\,\frac{1+q}{\sqrt{q}} & 4\,\frac{(1+q)^3}{q^{3/2}} & 4\,\frac{1+q}{\sqrt{q}} & 4\,\frac{(1+q)^3}{q^{3/2}} \\
2\,\frac{(1+q)^4}{q^2} & 2\,\frac{\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)(1+q)^2}{q^3} & 2\,\frac{3\,q^3+1+3\,q+8\,q^2+q^4}{q^2} & 2\,\frac{\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)(1+q)^2}{q^3} \\
2\,\frac{(1+q)^3}{q^{3/2}} & \frac{\left(q^2+6\,q+1\right)(1+q)^3}{q^{5/2}} & \frac{(1+q)\left(q^2+6\,q+1\right)}{q^{3/2}} & 2\,\frac{(1+q)\left(3\,q^3+1+3\,q+8\,q^2+q^4\right)}{q^{5/2}} \\
4\,\frac{(1+q)^2}{q} & 4\,\frac{(1+q)^4}{q^2} & 4\,\frac{(1+q)^2}{q} & 2\,\frac{3\,q^4+6\,q^3+14\,q^2+6\,q+3}{q^2}
\end{array}
\right]
$$

Figure 1: Coefficients of the elements of W1 and W2 in the symmetrized Kazhdan-Lusztig basis, as computed in [GCT4]

and

$$x = disc^{1/2}.$$

Since $disc$ is not a square, x does not belong to $Q(q)$. Let $K = Q(q)[x]$ be the algebraic extension of $Q(q)$ obtained by adjoining x. It is shown in [GCT7] that $\mathcal{B}$ admits a complete Wederburn-structure decomposition over K, but not $Q(q)$. In what follows, we assume that $\mathcal{B}$ is defined over this base field K.

The nonstandard canonical bases $E(3)$ and $E^{opp}(3)$ of $\mathcal{B}$ were computed in MATLAB/Maple using the algorithm in Section 3. They are as follows.

Let U_i, $1 \leq i \leq 5$, be the K-span of the entries $u_i^1, u_i^{12}, u_i^{21}, u_i^2 \in \mathcal{B}$ of the matrix

$$u_i = \begin{bmatrix} u_i^1 & u_i^{12} \\ u_i^{21} & u_i^2 \end{bmatrix},$$

where u_1^1 is as specified in Figure 6, u_2^1 the element obtained from u_1^1 by substituting $-x$ for x, and u_3^1, u_4^1, u_5^1 as specified in Figures 7-9. Elements are specified in these figures by giving their nonzero coefficients in the $\{\mathcal{Q}_\sigma\}$ basis; the coefficient for $\mathcal{Q}_\sigma$ is shown in front of σ. Let u_i^2, $1 \leq i \leq 5$, be the element obtained from u_i^1 by interchanging $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let $u_i^{12} = u_i^1 \mathcal{Q}_2$, and $u_i^{21} = \mathcal{Q}_2 u_i^1$, for $1 \leq i \leq 5$. Let $u_0 = 1$ (this definition of u_0 is different from that in [GCT7]). Then u_0 and the entries of u_i form the canonical basis $E(3)$ of $\mathcal{B}_3^H(q)$. The left cells of $E(e)$ are $\{u_0\}$ and the columns of u_i. The right cells are $\{u_0\}$ and the rows of u_i. The representation supported by $\{u_0\}$ is the trivial one-dimensional representation; let us denote it by Σ. The representations supported by the columns or rows of u_i are two-dimensional representations of $\mathcal{B}$, distinct for each i; let us denote them by χ^i. The left cell $\{u_0\}$ is at the top of the $\leq_L$ partial order and the left cells corresponding to the columns U_i are at depth 1 from the top in this partial order (and mutually incomparable). The situation for the right cells is similar.

Let

$$v_i = \begin{bmatrix} v_i^1 & v_i^{12} \\ v_i^{21} & v_i^2 \end{bmatrix},$$

where v_i^α is obtained from u_i^α by substituting $\mathcal{P}_\sigma$ for $\mathcal{Q}_\sigma$ in the expression of u_i^α in the $\{Q_\sigma\}$ basis. Let v_0 be the element whose coefficients in the $\{\mathcal{P}_\sigma\}$ basis are as shown in Figures 10-11. Then v_0 and the elements of v_i form the opposite canonical basis $E^{opp}(3)$ of $\mathcal{B}_3^H(q)$. The left cells are $\{v_0\}$ and the columns of v_i. The right cells are $\{v_0\}$ and the rows of v_i. The left cell $\{v_0\}$ is at the bottom of the $\leq_L$ partial order and the left cells corresponding to the columns of v_i at height 1 from the bottom (and mutually incomparable). The situation for the right cells is similar.

Let Σ' be the trivial one-dimensional representation of the subalgebra $\mathcal{B}_2^H(q) \subset \mathcal{B} = \mathcal{B}_3^H(q)$ generated by $\mathcal{P}_2$. Let μ' be the other one-dimensional representation of $\mathcal{B}_2^H(q)$ that specializes to the signed one-dimensional representation of the symmetric group S_2 at $q = 1$. Then the nonstandard Gelfand-Tsetlin tableau $T(b)$'s associated with the basis elements b's of $E(3)$ are as follows.

If $b = u_0$, then $T(b) = [\Sigma, \Sigma']$. If $b = u_i^1$, then $T(b) = [\chi^i, \Sigma']$. If $b = u_i^{21}$, then $T(b) = [\chi^i, \mu']$. If $b = u_i^2$, then $T(b) = [\chi^i, \Sigma']$. If $b = u_i^{12}$, then $T(b) = [\chi^i, \mu']$.

The nonstandard Gelfand-Tsetlin tableau associated with the basis elements in $E^{opp}(r)$ are similar.

5.3.1 Violation of standard balance

Let $R(3)$ be the A_K-lattice generated by $E(3)$, $\bar{R}(3) = (R(3))^-$. Let $R^{opp}(3)$ and $\bar{R}^{opp}(3)$ be defined similarly. Then it turns out that the triple $(\mathcal{B}_{3,Q}^H, R(3), \bar{R}(3))$ associated with the canonical basis $E(3)$ is balanced, but the triple $(\mathcal{B}_{3,Q}^H, R^{opp}(3), \bar{R}^{opp}(3))$ associated with the opposite canonical basis $E^{opp}(3)$ is not balanced. Specifically, the fibre $\psi_T^{-1}(0) \neq \{0\}$ when $T = T(b)$ is the nonstandard tableau associated with $b = v_i^2$, for any i (it is zero for all other b's).

For example, with the help of computer it was found that the Q-module $\psi_T^{-1}(0)$, for $b = v_5^2, T = T(b)$, is generated by the two elements w and x specified in Figures 12-15, which give their nonzero cofficients in the $\{\mathcal{P}_\sigma\}$ basis. Clearly w belongs to the K_Q form $\mathcal{B}_{3,Q}^H$ since the coefficients belong to $Q[q, q^{-1}]$. It is $-$-invariant, since the coefficients are $-$-invariant. It can be verified that $w \in qR^{opp}(3)$. Specifically, it can be shown that

$$w = a_0 v_0 + c^1 v_5^1 + c^{12} v_5^{12} + c^2 v_5^2 + c^{21} v_5^{21},$$

where the coefficient vector $[a_0, c^1, c^{12}, c^2, c^{21}]$ is the following

$$\begin{bmatrix} -\frac{q^6}{(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+1)^2} \\ \frac{(q^4-q^2+1)(q^4+1)q}{(q^2+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)} \\ -\frac{q^6}{(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+1)^2} \\ -\frac{q^6}{(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+1)^2} \\ \frac{(q^4-q^2+1)(q^4+1)q}{(q^2+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)} \end{bmatrix}$$

This means $\psi_T(w) = 0$. It can be verified that w belongs to $\mathcal{B}^{\leq T}$. Similarly, x is a $-$-invariant element of $\mathcal{B}^H_{3,Q} \cap \mathcal{B}^{\leq T}$ that belongs to $qR^{opp}(3)$.

5.4 Nonstandard globalization via minimization of degree complexity

Thus each element in $\psi_T^{-1}(\bar{b})$, $b = v_5^2$, $\bar{b} = \psi(b)$, $T = T(b)$, is a linear combination of b, w and x. It easy to see from the explicit formulae in Figures 12-15 and Figure 9 that $b = v_5^2$ is the element of minimum degree complexity in $\psi_T^{-1}(\bar{b})$, where the degree complexity is defined internally as in Section 4. (Remember that v_5^2 is obtained from u_5^1 in Figure 9 by substituting $\mathcal{P}_i$ for $\mathcal{Q}_i$ and then interchanging $\mathcal{P}_1$ and $\mathcal{P}_2$.) The same is also true for all b's. This verifies the minimum degree conjecture (Conjecture 3.1); Conjecture 3.2 can also be verified similarly.

We could not use the external definition of degree complexity as in Section 3.3 here, since the smallest product of Hecke algebras containing $\mathcal{B}_3^H(q)$ is $Z = \mathcal{H}_3(q)^{\otimes 9}$ with dimension $10077696 = 6^9$. It is impossible to carry out symbolic computations in an algebra of this size in MATLAB/Maple.

$c(\gamma_1^1;\gamma_1^1;\gamma_1^1) = c(\gamma_1^{12};\gamma_1^1;\gamma_1^1) = -(1+q)^2 * (q^2+q+1) * (q-1)^2/q^3;$
$c(\gamma_2^1;\gamma_1^1;\gamma_{21}^1) = -(q^2+q+1) * (q-1)^2/q^2;$
$c(\gamma_{21}^1;\gamma_1^1;\gamma_{21}^1) = -(1+q)^2 * (q^2+q+1) * (q-1)^2/q^3;$
$c(\gamma_1^1;\gamma_1^1;\Sigma) = c(\gamma_1^2;\gamma_1^1;\Sigma) = c(\gamma_2^1;\gamma_1^1;\Sigma) = c(\gamma_2^2;\gamma_1^1;\Sigma) = 1/q * (1+q)^2;$
$c(\gamma_{12}^1;\gamma_{12}^1;\Sigma) = c(\gamma_{21}^1;\gamma_{12}^1;\Sigma) = (1+q)^2 * (6*q+5*q^2+5)/q^2;$
$c(\gamma_{21}^2;\gamma_{21}^2;\gamma_{21}^2) = (1+q^2)^2 * (q^2+q+1) * (q-1)^2/q^4;$
$c(\gamma_{21}^2;\gamma_{21}^2;\Sigma) = c(\gamma_{12}^2;\gamma_{21}^2;\Sigma) = (1+q)^2 * (2*q^2+q+1) * (q^2+q+2)/q^3.$

Figure 2: Multiplicative structural constants of the canonical basis of $\mathcal{B}_3^H(q)$ in the Kronecker case, $n = 2$, $r = 3$

10	4	3		
44	21	7		
22	17	7	1	
50	30	15	2	
20	12	4		
14	7	3		
44	31	14	5	
19	12	4	1	
6	5	3	1	
94	64	29	4	
88	65	28	7	
39	24	8	1	
80	45	17	2	
40	32	16	4	
28	21	10	3	
75	45	19	5	
38	31	16	5	1
11	8	4	1	
122	69	23	2	
62	49	23	5	

Figure 3: The vectors associated with distinct nonzero coefficients of $\Sigma \in \mathcal{B}_4^H(q)$ (ignoring a positive, unimodal factor)

126	92	47	12	2	
158	93	33	4		
153	93	35	7		
78	63	32	9	1	
160	125	62	19	2	
72	48	20	4		
150	120	64	24	5	
69	47	21	6	1	
22	19	12	5	1	
212	163	74	17		
244	191	92	25	2	
111	72	28	5		
102	71	33	9	1	
104	88	52	20	4	
100	85	52	22	6	1
30	23	13	5	1	
128	106	59	20	3	
316	251	126	37	4	
306	246	128	42	7	
141	95	41	10	1	

Figure 4: The vectors associated with distinct nonzero coefficients of $\Sigma \in \mathcal{B}_4^H(q)$, continued

144	120	68	24	4		
41	31	17	6	1		
344	213	79	12			
375	237	91	17			
162	117	59	19	3		
222	183	100	33	5		
204	173	104	42	10	1	
192	140	72	24	4		
60	53	36	18	6	1	
220	191	124	58	18	3	
264	188	92	28	4		
82	72	48	23	7	1	
280	244	160	76	24	4	
83	66	41	19	6	1	
750	612	328	108	17		
345	237	107	28	3		
324	279	176	78	22	3	
113	89	54	24	7	1	
106	96	71	42	19	6	1
528	452	280	120	32	4	

Figure 5: The vectors associated with distinct nonzero coefficients of $\Sigma \in \mathcal{B}_4^H(q)$ (continued)

σ	$Coefficient$
1	$1/2\,(q^4-q^2+1)^2(q^8-q^6+q^4-q^2+1)^2(q^4+1)^2(q^2+1)^4$ $\times(x+3\,q^{28}+4\,q^{24}-2\,q^{22}+10\,q^{20}-2\,q^{18}+4\,q^{16}+3\,q^{12})/q^{40}$
121	$-1/2\,(q^2+1)^2(2\,q^{18}-295\,q^{28}-516\,q^{36}+x+210\,q^{26}+3\,q^{56}-3\,q^{54}+47\,q^{46}+9\,q^{52}-q^{48}$ $+2\,q^{50}-84\,q^{24}-295\,q^{40}+604\,q^{34}+462\,q^{30}-xq^2+47\,q^{22}-9\,q^{20}x+19\,q^{10}x-q^{26}x+q^{28}x-3\,q^{14}$ $+q^{24}x+4\,q^{22}x+30\,q^{14}x+462\,q^{38}-516\,q^{32}+19\,q^{18}x+210\,q^{42}-q^{20}+9\,q^{16}-24\,q^{16}x-24\,q^{12}x$ $-9\,q^8x+4\,q^6x+q^4x-84\,q^{44}+3\,q^{12})/q^{36}$
12121	$1/2\,(q^{18}-2\,q^{28}+22\,q^{36}+x+45\,q^{26}+2\,q^{46}+3\,q^{48}+22\,q^{24}+24\,q^{40}+45\,q^{34}+92\,q^{30}+18\,q^{22}+q^{20}x$ $+6\,q^{10}x+2\,q^{14}+q^{14}x+18\,q^{38}-2\,q^{32}+q^{42}+24\,q^{20}+9\,q^{16}+q^{16}x+q^6x+q^4x+9\,q^{44}+3\,q^{12})/q^{30}$
1212121	$-1/2\,(22\,q^{20}+6\,q^{16}+6\,q^{24}+2\,q^{26}+2\,q^{14}+2\,q^{30}+2\,q^{10}+3\,q^{28}-2\,q^{22}-2\,q^{18}+3\,q^{12}+x)/q^{20}$
121212121	1

Figure 6: Coefficients of u_1^1

σ	$Coefficient$
1	$(q^8 - q^6 + q^4 - q^2 + 1)^2 (q^{12} + q^{10} + 2\,q^8 + 2\,q^4 + q^2 + 1)^2 (q^2 + 1)^4 (q^4 - q^2 + 1)^4 / q^{32}$
121	$-(1 - 4\,q^{10} + 14\,q^8 - 30\,q^{14} + 44\,q^{28} + 73\,q^{16} + 3\,q^2 + 14\,q^{32} - 30\,q^{26} + 73\,q^{24} + 3\,q^{38}$ $+102\,q^{20} - 53\,q^{18} + q^{40} + 44\,q^{12} - 53\,q^{22} - 4\,q^{30} + 5\,q^4 + 5\,q^{36})(q^2 + 1)^2 (q^4 - q^2 + 1)^2 / q^{26}$
12121	$(3 + 72\,q^{18} + 14\,q^{28} + 3\,q^{36} + 20\,q^{26} + 10\,q^{24} + 2\,q^{34} + 2\,q^{30} + 36\,q^{22} + 14\,q^8 + 7\,q^4$ $+2\,q^2 + 7\,q^{32} + 2\,q^6 - 10\,q^{20} - 10\,q^{16} + 10\,q^{12} + 20\,q^{10} + 36\,q^{14}) / q^{18}$
1212121	$-(1 - 2\,q^{12} + 14\,q^{10} - 2\,q^8 + q^4 + 3\,q^2 + 4\,q^6 + q^{20} + 4\,q^{14} + 3\,q^{18} + q^{16}) / q^{10}$
121212121	1

Figure 7: Coefficients of u_3^1

σ	$Coefficient$
1	$(q^2+1)^2(q^4-q^2+1)^2(q^4+1)^2(q^8-q^6+q^4-q^2+1)^2(q^{12}+q^{10}+2\,q^8+2\,q^4+q^2+1)^2/q^{30}$
121	$-(1+75\,q^{18}-49\,q^{28}+42\,q^{36}+206\,q^{26}-q^{46}+q^{52}+7\,q^{48}-49\,q^{24}+40\,q^{40}+75\,q^{34}+158\,q^{30}+158\,q^{22}$ $+22\,q^8+7\,q^4+17\,q^{14}+17\,q^{38}+q^{32}-q^6+q^{20}+42\,q^{16}+22\,q^{44}+40\,q^{12})q^{26}$
12121	$(3+80\,q^{18}+10\,q^{28}+3\,q^{36}+26\,q^{26}-3\,q^{24}+q^{34}+5\,q^{30}+52\,q^{22}+10\,q^8+5\,q^4+52\,q^{14}+q^2+5\,q^{32}$ $+5\,q^6-19\,q^{20}-19\,q^{16}-3\,q^{12}+26\,q^{10})/q^{18}$
1212121	$-(1-2\,q^{12}+14\,q^{10}-2\,q^8+3\,q^2+5\,q^6+q^{20}+5\,q^{14}+3\,q^{18})/q^{10}$
121212121	1

Figure 8: Coefficients of u_4^1

σ	$Coefficient$
1	$(q^2+1)^2(q^4+1)^2(q^{12}+q^{10}+2\,q^8+2\,q^4+q^2+1)^2(q^4-q^2+1)^4/q^{26}$
121	$-(q^{36}+3\,q^{34}+10\,q^{32}+19\,q^{30}+33\,q^{28}+53\,q^{26}+64\,q^{24}+91\,q^{22}+84\,q^{20}+116\,q^{18}+84\,q^{16}+91\,q^{14}$ $+64\,q^{12}+53\,q^{10}+33\,q^8+19\,q^6+10\,q^4+3\,q^2+1)(q^4-q^2+1)^2/q^{22}$
12121	$(80\,q^{16}+3\,q^{26}+26\,q^{24}+4\,q^{22}+q^{32}+7\,q^{28}+50\,q^{20}+3\,q^6+3\,q^2+50\,q^{12}+1+3\,q^{30}+7\,q^4-14\,q^{18}$ $-14\,q^{14}+4\,q^{10}+26\,q^8)/q^{16}$
1212121	$-(3+5\,q^{12}-2\,q^{10}+14\,q^8+5\,q^4+q^2-2\,q^6+q^{14}+3\,q^{16})/q^8$
121212121	1

Figure 9: Coefficients of u_5^1

σ	$Coefficient$
$\emptyset$	$(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6-q^3+1)(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $\times(q^6+q^5+q^4+q^3+q^2+q+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+q+1)^2(q^2-q+1)^2$ $\times(q^4+1)^2(q-1)^4(q+1)^4(q^2+1)^4(q^4-q^2+1)^5/q^{46}$
2	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})/q^{41}$
1	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})/q^{41}$
12	$(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}$ $+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}$ $-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})(q^2+1)^2(q^4-q^2+1)^2/q^{36}$
21	$(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}$ $+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}$ $-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})(q^2+1)^2(q^4-q^2+1)^2/q^{36}$
212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}$ $-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}$ $+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{31}$
121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}$ $-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}$ $-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{31}$
1212	$(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}$ $+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{26}$
2121	$(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}$ $-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{26}$

Figure 10: First nine coefficients of v_0 in $\{\mathcal{P}_\sigma\}$ basis

σ	$Coefficient$
21212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8-q^6+q^4+1)(q^8+q^4-q^2+1)/q^{21}$
12121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8-q^6+q^4+1)(q^8+q^4-q^2+1)/q^{21}$
121212	$(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{16}$
212121	$(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{16}$
2121212	$(q^2+1)(q^4-q^2+1)(q^4+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
1212121	$(q^2+1)(q^4-q^2+1)(q^4+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
12121212	$-(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^8$
21212121	$-(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^8$
212121212	$-(q^2+1)(q^4-q^2+1)(q^4+1)/q^5$
121212121	$-(q^2+1)(q^4-q^2+1)(q^4+1)/q^5$
1212121212	1
2121212121	1

Figure 11: Last twelve coefficients of v_0 in $\{\mathcal{P}_\sigma\}$ basis

σ	$Coefficient$
$\emptyset$	$((q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)(q^6-q^3+1)(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $(q^6+q^5+q^4+q^3+q^2+q+1)(q^4+1)^2(q^2-q+1)^2(1+q^2+q)^2(q-1)^4(q+1)^4(q^2+1)^2$ $(q^4-q^2+1)^5)/q^{40}$
2	$-((q^2+1)(q^4+1)(q^4-q^2+1)^3(1-q^{20}-3\,q^{38}+q^{44}-3\,q^{14}+q^4-q^2+q^{12}+q^8-q^6-2\,q^{10}+q^{40}$ $-14\,q^{26}-8\,q^{34}-q^{32}-3\,q^{24}-10\,q^{22}-3\,q^{28}+q^{48}-10\,q^{30}+q^{52}-8\,q^{18}-q^{46}-q^{50}-2\,q^{42}))/q^{35}$
1	$-((q^2+1)(q^4+1)(q^4-q^2+1)^3(1-q^{20}-3\,q^{38}+q^{44}-3\,q^{14}+q^4-q^2+q^{12}+q^8-q^6-2\,q^{10}$ $+q^{40}-14\,q^{26}-8\,q^{34}-q^{32}-3\,q^{24}-10\,q^{22}-3\,q^{28}+q^{48}-10\,q^{30}+q^{52}-8\,q^{18}-q^{46}-q^{50}-2\,q^{42}))/q^{35}$
12	$((q^4-q^2+1)^2(1-q^{20}-3\,q^{38}+q^{44}-3\,q^{14}+q^4-q^2+q^{12}+q^8-q^6-2\,q^{10}+q^{40}-14\,q^{26}-8\,q^{34}$ $-q^{32}-3\,q^{24}-10\,q^{22}-3\,q^{28}+q^{48}-10\,q^{30}+q^{52}-8\,q^{18}-q^{46}-q^{50}-2\,q^{42}))/q^{30}$
21	$((q^4-q^2+1)^2(1-q^{20}-3\,q^{38}+q^{44}-3\,q^{14}+q^4-q^2+q^{12}+q^8-q^6-2\,q^{10}+q^{40}-14\,q^{26}-8\,q^{34}-q^{32}$ $-3\,q^{24}-10\,q^{22}-3\,q^{28}+q^{48}-10\,q^{30}+q^{52}-8\,q^{18}-q^{46}-q^{50}-2\,q^{42}))/q^{30}$
212	$-((q^2+1)(q^4+1)(q^4-q^2+1)(q^{40}-3\,q^{38}+4\,q^{36}-4\,q^{34}+7\,q^{32}-10\,q^{30}+19\,q^{28}-13\,q^{26}+25\,q^{24}$ $-22\,q^{22}+40\,q^{20}-22\,q^{18}+25\,q^{16}-13\,q^{14}+19\,q^{12}-10\,q^{10}+7\,q^8-4\,q^6+4\,q^4-3\,q^2+1))/q^{25}$
121	$-((q^2+1)(q^4+1)(q^4-q^2+1)(q^{40}-3\,q^{38}+4\,q^{36}-4\,q^{34}+7\,q^{32}-10\,q^{30}+19\,q^{28}-13\,q^{26}+25\,q^{24}$ $-22\,q^{22}+40\,q^{20}-22\,q^{18}+25\,q^{16}-13\,q^{14}+19\,q^{12}-10\,q^{10}+7\,q^8-4\,q^6+4\,q^4-3\,q^2+1))/q^{25}$
1212	$((q^{40}-3\,q^{38}+4\,q^{36}-4\,q^{34}+7\,q^{32}-10\,q^{30}+19\,q^{28}-13\,q^{26}+25\,q^{24}-22\,q^{22}+40\,q^{20}-22\,q^{18}+25\,q^{16}$ $-13\,q^{14}+19\,q^{12}-10\,q^{10}+7\,q^8-4\,q^6+4\,q^4-3\,q^2+1))/q^{20}$

Figure 12: A $-$-invariant element w in $qR^{opp}(3)$: the first eight coefficients

σ	$Coefficient$
2121	$(q^{40} - 3\,q^{38} + 4\,q^{36} - 4\,q^{34} + 7\,q^{32} - 10\,q^{30} + 19\,q^{28} - 13\,q^{26} + 25\,q^{24} - 22\,q^{22} + 40\,q^{20} - 22\,q^{18} + 25\,q^{16}$ $-13\,q^{14} + 19\,q^{12} - 10\,q^{10} + 7\,q^{8} - 4\,q^{6} + 4\,q^{4} - 3\,q^{2} + 1)/q^{20}$
21212	$-((q^2+1)(q^4+1)(q^4-q^2+1)(q^{20} - 3\,q^{18} + q^{16} - 3\,q^{14} + 3\,q^{12} - 10\,q^{10} + 3\,q^{8} - 3\,q^{6} + q^{4} - 3\,q^{2} + 1))/q^{15}$
12121	$-((q^2+1)(q^4+1)(q^4-q^2+1)(q^{20} - 3\,q^{18} + q^{16} - 3\,q^{14} + 3\,q^{12} - 10\,q^{10} + 3\,q^{8} - 3\,q^{6} + q^{4} - 3\,q^{2} + 1))/q^{15}$
121212	$(q^{20} - 3\,q^{18} + q^{16} - 3\,q^{14} + 3\,q^{12} - 10\,q^{10} + 3\,q^{8} - 3\,q^{6} + q^{4} - 3\,q^{2} + 1)/q^{10}$
212121	$(q^{20} - 3\,q^{18} + q^{16} - 3\,q^{14} + 3\,q^{12} - 10\,q^{10} + 3\,q^{8} - 3\,q^{6} + q^{4} - 3\,q^{2} + 1)/q^{10}$
2121212	$-((q^2+1)(q^4+1)(q^4-q^2+1))/q^5$
1212121	$-((q^2+1)(q^4+1)(q^4-q^2+1))/q^5$
12121212	1
21212121	1

Figure 13: A $-$-invariant element w in $qR^{opp}(3)$: the last nine coefficients

σ	$Coefficient$
$\emptyset$	$((q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)(q^6-q^3+1)(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $(q^6+q^5+q^4+q^3+q^2+q+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^4+1)^2(q^2-q+1)^2(q^2+q+1)^2$ $(q-1)^4(q+1)^4(q^2+1)^3(q^4-q^2+1)^5)/q^{45}$
2	$-((q^2+1)^2(q^4-q^2+1)^3(q^4+1)(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-12\,q^{46}+13\,q^{12}-12\,q^{10}+q^{24}$ $-10\,q^{18}+13\,q^{16}-12\,q^{14}+16\,q^{26}+13\,q^{48}+13\,q^{44}+28\,q^{30}-10\,q^{42}+11\,q^{52}+2\,q^{60}-9\,q^{54}+7\,q^{56}$ $-4\,q^{58}+12\,q^{20}-12\,q^{50}-11\,q^{28}+16\,q^{34}+q^{22}+q^{36}-11\,q^{32}+q^{38}+12\,q^{40}))/q^{40}$
1	$-((q^2+1)^2(q^4-q^2+1)^3(q^4+1)(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-12\,q^{46}+13\,q^{12}-12\,q^{10}+q^{24}$ $-10\,q^{18}+13\,q^{16}-12\,q^{14}+16\,q^{26}+13\,q^{48}+13\,q^{44}+28\,q^{30}-10\,q^{42}+11\,q^{52}+2\,q^{60}-9\,q^{54}+7\,q^{56}-4\,q^{58}$ $+12\,q^{20}-12\,q^{50}-11\,q^{28}+16\,q^{34}+q^{22}+q^{36}-11\,q^{32}+q^{38}+12\,q^{40}))/q^{40}$
12	$((q^2+1)(q^4-q^2+1)^2(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(1+q^4-q^2-q^6+q^8$ $-q^{46}+q^{12}-2\,q^{10}-3\,q^{24}-8\,q^{18}-3\,q^{14}-14\,q^{26}+q^{48}+q^{44}-10\,q^{30}-2\,q^{42}+q^{52}-q^{20}-q^{50}$ $-3\,q^{28}-8\,q^{34}-10\,q^{22}-q^{32}-3\,q^{38}+q^{40}))/q^{35}$
21	$((q^2+1)(q^4-q^2+1)^2(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(1+q^4-q^2-q^6+q^8-q^{46}+q^{12}-2\,q^{10}$ $-3\,q^{24}-8\,q^{18}-3\,q^{14}-14\,q^{26}+q^{48}+q^{44}-10\,q^{30}-2\,q^{42}+q^{52}-q^{20}-q^{50}-3\,q^{28}-8\,q^{34}-10\,q^{22}$ $-q^{32}-3\,q^{38}+q^{40}))/q^{35}$
212	$-((q^4-q^2+1)(q^4+1)(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-2\,q^{46}-3\,q^{12}-9\,q^{10}+53\,q^{24}-48\,q^{18}+q^{16}$ $-13\,q^{14}-110\,q^{26}+2\,q^{48}-2\,q^{44}-77\,q^{30}-9\,q^{42}+2\,q^{52}+27\,q^{20}-4\,q^{50}+53\,q^{28}-48\,q^{34}-77\,q^{22}$ $+q^{36}+27\,q^{32}-13\,q^{38}-3\,q^{40}))/q^{30}$
121	$-((q^4-q^2+1)(q^4+1)(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-2\,q^{46}-3\,q^{12}-9\,q^{10}+53\,q^{24}-48\,q^{18}+q^{16}$ $-13\,q^{14}-110\,q^{26}+2\,q^{48}-2\,q^{44}-77\,q^{30}-9\,q^{42}+2\,q^{52}+27\,q^{20}-4\,q^{50}+53\,q^{28}-48\,q^{34}-77\,q^{22}$ $+q^{36}+27\,q^{32}-13\,q^{38}-3\,q^{40}))/q^{30}$
1212	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^{40}-3\,q^{38}+4\,q^{36}-4\,q^{34}+7\,q^{32}-10\,q^{30}+19\,q^{28}-13\,q^{26}$ $+25\,q^{24}-22\,q^{22}+40\,q^{20}-22\,q^{18}+25\,q^{16}-13\,q^{14}+19\,q^{12}-10\,q^{10}+7\,q^8-4\,q^6+4\,q^4-3\,q^2+1)(q^2+1))/q^{25}$

Figure 14: A $-$-invariant element x in $qR^{opp}(3)$: the first eight coefficients

σ	$Coefficient$
2121	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^{40}-3\,q^{38}+4\,q^{36}-4\,q^{34}+7\,q^{32}-10\,q^{30}+19\,q^{28}-13\,q^{26}+25\,q^{24}$ $-22\,q^{22}+40\,q^{20}-22\,q^{18}+25\,q^{16}-13\,q^{14}+19\,q^{12}-10\,q^{10}+7\,q^8-4\,q^6+4\,q^4-3\,q^2+1)(q^2+1))/q^{25}$
21212	$-((q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1))/q^{20}$
12121	$-((q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1))/q^{20}$
121212	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^{20}-3\,q^{18}+q^{16}-3\,q^{14}+3\,q^{12}-10\,q^{10}+3\,q^8-3\,q^6+q^4-3\,q^2+1)$ $(q^2+1))/q^{15}$
212121	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^{20}-3\,q^{18}+q^{16}-3\,q^{14}+3\,q^{12}-10\,q^{10}+3\,q^8-3\,q^6+q^4-3\,q^2+1)$ $(q^2+1))//q^{15}$
2121212	$((q^4+1)(q^4-q^2+1)(-3\,q^6-3\,q^{10}+12\,q^8-q^2+3+3\,q^4+3\,q^{16}-q^{14}+3\,q^{12}))/q^{12}$
1212121	$((q^4+1)(q^4-q^2+1)(-3\,q^6-3\,q^{10}+12\,q^8-q^2+3+3\,q^4+3\,q^{16}-q^{14}+3\,q^{12}))/q^{12}$
12121212	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+1))/q^5$
21212121	$((2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+1))/q^5$
212121212	$-((q^4+1)(q^4-q^2+1))/q^4$
121212121	$-((q^4+1)(q^4-q^2+1))/q^4$

Figure 15: A $-$-invariant element x in $qR^{opp}(3)$: the last eleven coefficients

5.4.1 Nonstandard positivity

Now we shall describe evidence for Conjecture 3.6 in the case under consideration.

So far we are assuming that $\mathcal{P}_i$'s are as defined in the begining of Section 7.1 in [GCT7]. But as explained towards the end of that section, the actual $\mathcal{P}_i$'s differ from these (chosen for convenience and simplicity) by a positive, unimodal factor $\hat{f}_p(q) \in Q[q, q^{-1}]$ given there. As it turns this does not matter in the calculations so far, but for rescaling of the picture, but it does in the study of the positivity properties below. Rescaling of $\mathcal{P}_i$ by $\hat{f}_p(q)$ implies that each structural coefficient or constant $c(q)$ computed so far has to be multiplied by an appropriate power of $\hat{f}_p(q)^{m(c)}$, where $m(c)$ is a nonnegative integer depending on c. In what follows, it is implicitly assumed that each c computed so far has been rescaled by a suitable power $\hat{f}_p(q)^{n(c)}$, where $n(c) \leq m(c)$ is the smallest nonnegative integer chosen so that the (nonstandard) positivity property of the coefficients of c becomes apparent The picture remains the same even if we were to multiply by $\hat{f}_p(q)^{m(c)}$, but we choose $n(c)$ as small as possible to keep the the degrees of the polynomials from blowing up.

Figures 16-20 show the vectors associated with the nonzero coefficients of v_0 in the $\{\mathcal{P}_\sigma\}$ basis. The vector (as defined in the begining of Section 5) for the coefficient corresponding to each σ is obtained by concatenating the rows in front of that σ. Figure 21 shows the vectors associated with the traces of the nonzero coefficients of v_1^1 and Figures 22-23 show their norms. Here the trace and norm of an element in the algerbraic extension K of $Q(q)$ is defined in the usual fashion as the sum and product of its images under the Frobenius automorphisms of K over $Q(q)$; they are coefficients of the minimal polynomial of the element. It can be seen that all vectors in Figures 16-23 are positive and nonincreasing, except for the vectors associated with v_0 for $\sigma = 212, 121, 1212, 2121$, which are almost positive and nonincreasing. It was verified with the help of computer that the vectors associated with the coefficients of other canonical basis elements are similarly either positive and nonincreasing or almost positive and nonincreasing. This is in accordance with Conjectures 3.5-3.6.

Figures 24-35 show vectors associated with a few multiplicative constants, taking norms and traces whenever necessary; the coefficient of bb' with respect to the basis element b'' is denoted by $c(b, b'; b'')$. Again it can be seen that these vectors are positive and nonincreasing, except a few, which are almost positive and nonincreasing. It was verified with the help of computer that the picture is the same for other multiplicative constants as well. This too is in accordance with Conjectures 3.5-3.6.

σ	vector						
$\emptyset$	34116640	34028832	33766665	33333910	32736719	31983492	31084702
	30052720	28901524	27646408	26303647	24890162	23423208	21920062
	20397697	18872456	17359791	15874002	14428069	13033484	11700135
	10436190	9248100	8140602	7116788	6178174	5324820	4555450
	3867635	3257976	2722277	2255718	1853025	1508640	1216881
	972102	768790	601662	465735	356396	269443	201126
	148131	107564	76935	54142	37439	25404	16892
	10988	6976	4308	2576	1484	821	434
	217	100	40	12	2		
2	13180	13086	12992	12744	12496	12124	11752
	11225	10698	10112	9526	8890	8254	7584
	6914	6294	5674	5083	4492	3979	3466
	3036	2606	2256	1906	1638	1370	1178
	986	840	694	603	512	450	388
	335	282	259	236	206	176	153
	130	116	102	85	68	54	40
	34	28	21	14	9	4	4
	4	2					

Figure 16: The vectors associated with the coefficients of v_0

σ	vector						
1	13180	13086	12992	12744	12496	12124	11752
	11225	10698	10112	9526	8890	8254	7584
	6914	6294	5674	5083	4492	3979	3466
	3036	2606	2256	1906	1638	1370	1178
	986	840	694	603	512	450	388
	335	282	259	236	206	176	153
	130	116	102	85	68	54	40
	34	28	21	14	9	4	4
	4	2					
12	3432	3432	3379	3326	3242	3158	3033
	2908	2744	2580	2417	2254	2069	1884
	1709	1534	1371	1208	1062	916	797
	678	581	484	411	338	287	236
	202	168	143	118	108	98	84
	70	65	60	56	52	43	34
	30	26	23	20	15	10	7
	4	4	4	2			

Figure 17: The vectors associated with the coefficients of v_0 (continued)

σ	vector								
21	3432	3432	3379	3326	3242	3158	3033	2908	2744
	2580	2417	2254	2069	1884	1709	1534	1371	1208
	1062	916	797	678	581	484	411	338	287
	236	202	168	143	118	108	98	84	70
	65	60	56	52	43	34	30	26	23
	20	15	10	7	4	4	4	2	
212	13352	13285	13218	12957	12696	12341	11986	11462	10938
	10358	9778	9131	8484	7812	7140	6498	5856	5240
	4624	4088	3552	3079	2606	2236	1866	1552	1238
	1026	814	646	478	373	268	198	128	93
	58	35	12	4	−4	−4	−4	−4	−4
	−4	−4	−2						
121	13352	13285	13218	12957	12696	12341	11986	11462	10938
	10358	9778	9131	8484	7812	7140	6498	5856	5240
	4624	4088	3552	3079	2606	2236	1866	1552	1238
	1026	814	646	478	373	268	198	128	93
	58	35	12	4	−4	−4	−4	−4	−4
	−4	−4	−2						
1212	3472	3472	3427	3382	3293	3204	3093	2982	2810
	2638	2483	2328	2132	1936	1772	1608	1423	1238
	1098	958	820	682	587	492	398	304	249
	194	153	112	85	58	37	16	10	4
	2	0	−2	−4	−4	−4	−2		
2121	3472	3472	3427	3382	3293	3204	3093	2982	2810
	2638	2483	2328	2132	1936	1772	1608	1423	1238
	1098	958	820	682	587	492	398	304	249
	194	153	112	85	58	37	16	10	4
	2	0	−2	−4	−4	−4	−2		

Figure 18: The vectors associated with the coefficients of v_0 (continued)

σ	vector								
21212	5496	5453	5410	5286	5162	5008	4854	4600	4346
	4068	3790	3497	3204	2894	2584	2295	2006	1749
	1492	1280	1068	889	710	586	462	366	270
	215	160	117	74	56	38	27	16	11
	6	3							
12121	5496	5453	5410	5286	5162	5008	4854	4600	4346
	4068	3790	3497	3204	2894	2584	2295	2006	1749
	1492	1280	1068	889	710	586	462	366	270
	215	160	117	74	56	38	27	16	11
	6	3							
121212	1434	1434	1406	1378	1346	1314	1267	1220	1128
	1036	961	886	799	712	633	554	470	386
	334	282	231	180	146	112	82	52	42
	32	24	16	11	6	3			
212121	1434	1434	1406	1378	1346	1314	1267	1220	1128
	1036	961	886	799	712	633	554	470	386
	334	282	231	180	146	112	82	52	42
	32	24	16	11	6	3			
2121212	1004	992	980	957	934	908	882	829	776
	716	656	597	538	472	406	346	286	240
	194	158	122	94	66	51	36	26	16
	11	6	3						

Figure 19: The vectors associated with the coefficients of v_0 (continued)

σ	vector								
1212121	1004	992	980	957	934	908	882	829	776
	716	656	597	538	472	406	346	286	240
	194	158	122	94	66	51	36	26	16
	11	6	3						
12121212	258	258	252	246	245	244	237	230	208
	186	168	150	132	114	95	76	60	44
	37	30	23	16	11	6	3		
21212121	258	258	252	246	245	244	237	230	208
	186	168	150	132	114	95	76	60	44
	37	30	23	16	11	6	3		
212121212	68	67	66	65	64	63	62	58	54
	49	44	39	34	28	22	17	12	9
	6	4	2	1					
121212121	68	67	66	65	64	63	62	58	54
	49	44	39	34	28	22	17	12	9
	6	4	2	1					

Figure 20: The vectors associated with the coefficients of v_0 (continued)

σ	vector								
1	15864	15864	15680	15496	15160	14824	14334	13844	13228
	12612	11932	11252	10505	9758	9008	8258	7534	6810
	6126	5442	4834	4226	3702	3178	2738	2298	1948
	1598	1333	1068	868	668	534	400	310	220
	165	110	80	50	34	18	12	6	3
121	14690	14690	14484	14278	13930	13582	13065	12548	11880
	11212	10509	9806	9032	8258	7498	6738	6031	5324
	4684	4044	3498	2952	2515	2078	1729	1380	1124
	868	698	528	405	282	212	142	101	60
	42	24	15	6	3				
12121	4974	4974	4886	4798	4683	4568	4380	4192	3914
	3636	3367	3098	2803	2508	2217	1926	1669	1412
	1205	998	825	652	529	406	316	226	172
	118	89	60	41	22	14	6	3	
1212121	708	708	694	680	673	666	642	618	565
	512	464	416	365	314	262	210	170	130
	106	82	62	42	30	18	11	4	2

Figure 21: The vectors associated with the traces of the coefficients of v_1^1

σ	vector						
1	942062408	940617136	936306916	929157344	919242745	906637444	891460752
	873831980	853909410	831851324	807845688	782080468	754764061	726104864
	696321516	665632656	634256120	602409744	570301452	538139168	506112609
	474411492	443199968	412642188	382872865	354026712	326205988	299512952
	274016717	249786396	226859750	205274540	185039618	166163836	148631230
	132425836	117511844	103853444	91399832	80100204	69893842	60720028
	52513000	45206996	38734986	33029940	28027078	23661620	19873490
	16602612	13795174	11397364	9362691	7644664	6204292	5002584
	4007841	3188364	2519160	1975236	1537471	1186744	908860
	689624	518854	386368	285084	207920	150131	106972
	75388	52324	35881	24160	16048	10432	6681
	4164	2550	1508	874	484	262	132
	64	28	12	4	1		
121	835471628	833946584	829404381	821877948	811459617	798241720	782371271
	763995284	743308085	720504000	695809491	669451020	641674579	612726160
	582858876	552325840	521370802	490237512	459149620	428330776	397976239
	368281268	339402663	311497224	284682092	259074408	234750212	211785544
	190216267	170078244	151372598	134100452	118233841	103744800	90582172
	78694800	68016456	58480912	50013272	42538640	35978861	30255780
	25293821	21017408	17356415	14240716	11608322	9397244	7554889
	6028664	4775532	3752456	2925473	2260620	1732150	1314316
	988221	734968	541214	393616	283165	200852	140775
	97032	65989	44012	28926	18556	11701	7160
	4299	2484	1405	752	391	188	88
	36	14	4	1			

Figure 22: The vectors associated with the norms of the coefficients of v_1^1

12121	100225178	100001236	99336179	98236776	96719406	94800448	92504960
	89858000	86893481	83645316	80151751	76451032	72582611	68585940
	64501436	60369516	56227388	52112260	48055958	44090308	40241933
	36537456	32995876	29636192	26469934	23508632	20757113	18220204
	15896382	13784124	11876674	10167276	8645972	7302804	6125315
	5101048	4216788	3459320	2815838	2273536	1820593	1445188
	1137127	886216	684162	522672	395267	295468	218449
	159384	114887	81572	57144	39308	26622	17644
	11492	7284	4521	2704	1573	868	461
	224	102	40	15	4	1	
1212121	2212002	2205476	2186350	2155076	2112378	2058980	1995675
	1923256	1842975	1756084	1663849	1567536	1468102	1366504
	1263855	1161268	1059834	960644	864459	772040	684050
	601152	523831	452572	387494	328716	276117	229576
	188839	153652	123592	98236	77108	59732	45640
	34364	25488	18596	13346	9396	6493	4384
	2894	1848	1142	672	377	196	95
	40	15	4	1			

Figure 23: The vectors associated with the norms of the coefficients of v_1^1 (continued)

9846	9820	9750	9644	9501	9320	9093
8812	8493	8152	7779	7364	6917	6448
5966	5480	4989	4492	4001	3528	3080
2664	2274	1904	1569	1284	1038	820
631	472	346	256	186	120	69
44	31	16	4			

Figure 24: The vector for the multiplicative constant $c(v0, v_5^1; v0)$

13026	12964	12777	12464	12048	11552	10964
10272	9523	8764	7990	7196	6398	5612
4867	4192	3569	2980	2444	1980	1583
1248	967	732	539	384	267	188
131	80	42	24	16	8	2

Figure 25: The vector for the multiplicative constant $c(v0, v_4^1; v0)$

14180	14088	13832	13432	12894	12224	11448
10592	9682	8744	7800	6872	5976	5128
4342	3632	2998	2440	1956	1544	1202
928	706	520	374	272	198	136
88	56	36	24	16	8	2

Figure 26: The vector for the multiplicative constant $c(v0, v_3^1; v0)$

1356922	1356922	1341857	1326792	1297628	1268464
1227083	1185702	1133960	1082218	1023821	965424
902616	839808	776306	712804	650849	588894
531320	473746	421861	369976	325182	280388
243022	205656	175677	145698	122582	99466
82260	65054	52971	40888	32575	24262
19002	13742	10480	7218	5402	3586
2573	1560	1107	654	431	208
135	62	35	8	4	

Figure 27: The vector for the trace of the multiplicative constant $c(v0, v_1^1; v0)$

128607887512140	128408739182416	127813071267725
126826189536408	125456853005552	123717149426056
121622324577366	119190568251368	116442761279740
113402187959064	110094219510476	106545974210632
102785960441146	98843708903656	94749400393129
90533495522016	86226372383558	81857978142544
77457499613617	73053057887224	68671430858636
64337807515464	60075576278933	55906149694176
51848826289840	47920690427296	44136549385454
40508906927184	37047971392480	33761696363504
30655850888930	27734116255008	24998205742188
22448003806144	20081720818165	17896059499880
15886389332178	14046925218184	12370907081114
10850777077832	9478351722036	8244986211000
7141729026165	6159464877872	5289044837698
4521402501856	3847655805468	3259194107520
2747750779948	2305461534304	1924909254250

Figure 28: The vector for the norm of the multiplicative constant $c(v0, v_1^1; v0)$

1599156102128	1321763961652	1086804278768
888858554957	723010747256	584832826172
470364742664	376089969502	298907782312
236103258694	185315973800	144508153155
111933043504	86104032184	65765045520
49862540548	37519404368	28010898646
20742786784	15231640340	11087321280
7997549136	5714462144	4043023409
2831123144	1961213430	1343311944
909211769	607734400	400883734
260758832	167107927	105406152
65367982	39805384	23766532
13889944	7930388	4412904
2386644	1250232	631716
306184	141372	61592
25006	9272	3049
848	190	32
4		

Figure 29: The vector for the norm of the multiplicative constant $c(v0, v_1^1; v0)$ (continued)

612764	609940	601539	587774	568990	545654	518332
487666	454355	419136	382753	345926	309336	273610
239300	206862	176658	148958	123938	101678	82176
65362	51106	39226	29503	21696	15554	10828
7279	4686	2851	1604	800	316	52
−68	−101	−90	−64	−38	−18	−6
−1						

Figure 30: The vector for the multiplicative constant $c(v_5^1, v_5^1; v_5^1)$

9738	9694	9563	9348	9052	8676	8228
7732	7209	6658	6077	5484	4900	4332
3784	3268	2794	2360	1962	1604	1292
1028	809	626	471	344	246	172
116	76	50	32	17	6	1

Figure 31: The vector for the multiplicative constant $c(v_4^1, v_4^1; v_4^1)$

5.5 Experimental evidence for crystalization

Let $(L_\alpha^\rho, B_\alpha^\rho)$ be an upper crystal basis of $W_{q,\alpha}^\rho$ as in Conjecture 2.2. Since it is also a local crystal basis of $W_{q,\alpha}^\rho$ as an H_q-module, there is a crystal graph over B_α^ρ whose connected components correspond to irreducible H_q-submodules of $W_{q,\alpha}^\rho$. The elements of B_α^ρ that correspond to the highest weight nodes of these connected components are called the *highest weight crystal elements* of the upper crystal base $(L_\alpha^\rho, B_\alpha^\rho)$ with respect to H_q.

5.6 Kronecker problem: $n = 2, r = 3$

Consider again the special case of the Kronecker problem for $n = 2$ and $r = 3$ as in Section 5.1. Thus $H = Gl_2 \times Gl_2$, $G = Gl_4$ with H embedded diagonally, $X = X_q = V_q \otimes W_q$ is the standard four dimensional representation of H_q, where $V_q \cong W_q$ is the standard two representation of $GL_q(2)$.

81298	80886	79660	77650	74908	71510	67546
63110	58306	53254	48074	42870	37740	32786
28097	23732	19734	16144	12987	10258	7938
6010	4449	3212	2251	1526	999	628
374	208	107	50	20	6	1

Figure 32: The vector for the multiplicative constant $c(v_3^1, v_3^1; v_3^1)$

976672	974152	971632	954184	936736	915476
894216	860888	827560	791463	755366	713035
670704	627743	584782	540877	496972	454258
411544	373144	334744	298301	261858	232239
202620	175584	148548	128938	109328	91580
73832	62540	51248	41348	31448	25972
20496	15650	10804	8712	6620	4729
2838	2215	1592	986	380	312
244	113	-18	-9	0	-15
-30	-15				

Figure 33: The vector for the trace of the multiplicative constant $c(v_1^1, v_1^1; v_1^1)$

80255448992640	80138274925167	79787740098774
79206792680861	78400301248652	77374989989221
76139349486502	74703523453335	73079174468928
71279332267508	69318226392392	67211105097154
64974044241996	62623750255631	60177360010954
57652240221271	55065789702876	52435247936361
49777512793214	47108969815061	44445335040340
41801513256271	39191473712706	36628144773627
34123327701708	31687629536553	29330415837966
27059783595895	24882552858520	22804275396566
20829260005316	18960613784310	17200296928344
15549188413159	14007161465134	12573167695619
11245327257264	10021022340390	8896992772108
7869432671924	6934086054972	6086339325247
5321309766266	4633929430149	4019023248184
3471380269320	2985817677720	2557237529092
2180675978000	1851344837246	1564665615708

Figure 34: The vector for the norm of the multiplicative constant $c(v_1^1, v_1^1; v_1^1)$

1316296386448	1102151942468	918417720699
761557915658	628318302423	515724544156
421076738262	341940657408	276136144552
221723424848	176988052223	140424814566
110720869883	86738676456	67499233418
52165764220	40027907728	30486730148
23040841355	17273591498	12841266667
9462395940	6908272529	4994579014
3573971135	2529616304	1769689525
1222700770	833511103	559989972
370279871	240567010	153252303
95475892	57962133	34120478
19338377	10435196	5256321
2374094	864025	140844
-155240	-236220	-220838
-171968	-120043	-77166
-46045	-25460	-12974
-6040	-2518	-900
-255	-50	-5

Figure 35: The vector for the norm of the multiplicative constant $c(v_1^1, v_1^1; v_1^1)$ (continued)

Let

$$x_1 = v_1 \otimes w_1, x_2 = v_1 \otimes w_2, x_3 = v_2 \otimes w_1, x_4 = v_2 \otimes w_2,$$

be the standard basis of X_q. Let $b_1, \ldots, b_4$ be the corresponding standard crystal basis of $L(X_q)$. The irreducible representations of G_q^H that occur in $X_q^{\otimes 3}$ are:

1. $C_q^{H,3}(X)$, the 16-dimensional the degree-three component of the braided symmetric algebra of G_q^H,
2. $\wedge_q^{H,3}(X)$, the four dimensional degree three component of the braided exterior algebra of G_q^H,
3. two copies of the 16-dimensional G_q^H-representation $W_{q,(2,1),1}(X)$ defined in [GCT4] (it is denoted by $V_{q,(2,1),1}(X)$ there), and
4. two copies of the 4-dimensional representation $W_{q,(2,1),2}(X)$ of G_q^H as also defined there (it is called $V_{q,(2,1),2}(X)$ there).

Embeddings of the braided symmetric and exterior algebra components in $X_q^{\otimes 3}$ are uniquely defined. We denote their embedded images by $C_q^{H,3}(X)$ and $\wedge_q^{H,3}(X)$ again. We choose appropriate embeddings of $W_{q,(2,1),2}(X)$ and $W_{q,(2,1),2}(X)$ in $X_q^{\otimes 3}$ and denote them by the same symbols again. As $H_q = GL_q(V) \times GL_q(W)$-modules,

$$\begin{aligned}
\mathbb{C}_q^{H,3}(X) &= V_{q,(3)}(V) \otimes V_{q,(3)}(W) \oplus W_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W), \\
\wedge_q^{H,3}(X) &= V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W), \\
W_{q,(2,1),1}(X) &= V_{q,(2,1)}(V) \otimes V_{q,(3)}(W) \oplus V_{q,(3)}(V) \otimes V_{q,(2,1)}(W) \\
W_{q,(2,1),2}(X) &= V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W).
\end{aligned}$$

It was verified by computer that they have upper crystal bases as per Conjecture 2.2. The highest weight crystal elements with respect to H_q for these upper crystal bases, as shown separately for each module, are as follows; we denote the monomial basis element $b_{i_1} \otimes b_{i_2} \otimes b_{i_3}$ of $B(X_q^{\otimes 3})$ by $b_{i_1 i_2 i_3}$.

$\mathbb{C}_q^{H,3}(X)$: $\{b_{114} + b_{141}, b_{111}\}$.
$\wedge_q^{H,3}(X)$: $\{b_{123} + b_{132}\}$.
$W_{q,(2,1),1}(X)$: $\{b_{121}, b_{131}\}$.
$W_{q,(2,1),2}(X)$: $\{b_{114} - b_{141}\}$.

The highest weight crystal elements whose monomial support have size two correspond to the four dimensional H_q-module $V_{q,(2,1)}(V) \otimes V_{q,(2,1)}(W)$. The element b_{111} corresponds to the H_q-module $V_{q,(3)}(V) \otimes V_{q,(3)}(W)$, the element b_{121} to the H_q-module $V_{q,(3)}(V) \otimes V_{q,(2,1)}(W)$, and b_{131} to the H_q-module $V_{q,(2,1)}(V) \otimes V_{q,(3)}(W)$. Notice that not all highest weight crystal elements have monomial supports of size one as in the standard setting.

5.6.1 $H = sl_2$, $G = sl_4$

Now we consider the case when $H_q = Gl_q(2)$, X_q its four dimensional irreducible representation, and G_q^H as in Section 5.3. Let $W_0, \ldots, W_5$ be the irreducible representations of G_q^H occuring in $X_q^{\otimes 3}$ as defined in Section 6.1.2 of [GCT7], with $W_0 = \mathbb{C}_q^{H,3}[X]$.

As H_q-modules,

$$\begin{array}{rcl} W_0 & \cong & V_{q,(9)}(2) \oplus V_{q,(7,2)}(2), \\ W_1 & \cong & V_{q,(6,3)}(2), \\ W_2 & \cong & V_{q,(6,3)}(2), \\ W_3 & \cong & V_{q,(8,1)}(2), \\ W_4 & \cong & V_{q,(5,4)}(2), \\ W_5 & \cong & V_{q,(7,2)}(2), \end{array} \tag{19}$$

where $V_{q,\lambda}(n)$ denotes the q-Weyl module of $GL_q(n)$ corresponding to the partition λ. Their dimensions are $16, 4, 4, 8, 2$ and 6, respectively. Though W_1 and W_2 are isomorphic as H_q-modules, they are nonisomorphic as G_q^H-modules.

It was verified by computer that they–or rather their embeddings in $X_q^{\otimes 3}$–have upper crystal bases as per Conjecture 2.2. The highest weight crystal elements of of the embedding of $W_1, \ldots, W_5$ have monomial supports of size one. Let $b_{i_1 i_2 i_3} = b_{i_1} \otimes b_{i_2} \otimes b_{i_3}$ denote the monomial basis elements of $B(X_q^{\otimes 3})$. Then the highest weight crystal elements of the uniquely defined embedding of W_0 are b_{111} and $b = b_{113} + b_{131}$. The latter element b here has monomial support of size two, a phenomenon not seen in the standard setting. Nonzero coefficients of the element x in the lattice $L(\mathbb{C}_q^{3,H}(X))$

whose crystalization is b is shown in Figure 36, wherein $x_1, \ldots, x_4$ are the standard basis vectors of X_q.

6 Complexity theoretic properties of the canonical basis

In the standard setting, elements of the canonical basis of $V_{q,\lambda}$ are indexed (labelled) by semistandard tableau, for $H = GL(V)$, and by LS-paths [Li], for general semisimple H. And combinatorial analogues of Kashiwara's crystal operators [Li] on these labels can be computed efficiently [GCT6]. This is enough to imply a $\#P$-formula for the generalized Littlewood-Richardson coefficient though the canonical basis of $V_{q,\lambda}$ is hard to compute.

In the same spirit, it may be conjectured that the canonical basis of $\mathcal{O}(M_q^H(X)$ (or rather the set of its labels) has additional complexity theoretic properties (to be described in the full version), based on its cellular and refined sub-cellular decomposition (Conjecture 2.6), that imply a positive $\#P$-formula for the multiplicity n_π^α of the irreducible H_q-module $V_{q,\pi}$ in $W_{q,\alpha}$. This would solve the problem P1 in [GCT7].

Similarly, let m_λ^α denote the multiplicity of the Specht module S_λ of the symmetric group S_r corresponding to the partition λ in $\mathrm{Lim}_{q\to 1} T_{q,\alpha}$, considered as an S_r-module. It may be conjectured that the canonical basis $\mathcal{B}_r^H(q)$ (or rather the set of its labels) has similar additional complexity theoretic properties (to be described in the full version) based on its cellular and quasi-subcellular decompositions (Conjecture 3.3). This would imply a positive $\#P$-formula for the multiplicity m_λ^π, as needed in the problem P2 in [GCT7].

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$Monomial$	$Coefficient$
$x_1 \otimes x_1 \otimes x_3$	$-(q^4+1)^2(q^2+1)^4(q^4-q^2+1)^5(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$ $(q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)(2\,q^{20}-2\,q^{18}+q^{16}+q^{10}-q^8+q^6+q^4-q^2+1)$ $(q^2+q+1)^2(q^2-q+1)^2(q-1)^4(q+1)^4/q^{46}$
$x_1 \otimes x_2 \otimes x_2$	$-(q^2+1)^5(2\,q^6+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^2-q+1)^2(q^4+1)^2(q-1)^5(q+1)^5$ $(q^4-q^2+1)^5/q^{37}$
$x_1 \otimes x_3 \otimes x_1$	$-(q^2+1)^4(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$ $(q^6-q^5+q^4-q^3+q^2-q+1)(q^{16}+q^{14}-q^{12}+q^{10}+q^4-q^2+1)(q^4+1)^2(q^2-q+1)^2(q^2+q+1)^2$ $(q-1)^4(q+1)^4(q^4-q^2+1)^5/q^{46}$
$x_2 \otimes x_1 \otimes x_2$	$-(q^2+1)^5(2\,q^6+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $(q^4-q^3+q^2-q+1)(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^2-q+1)^2(q^4+1)^2(q-1)^5(q+1)^5$ $(q^4-q^2+1)^5/q^{40}$
$x_2 \otimes x_2 \otimes x_1$	$-(q^2+1)^5(2\,q^6+1)(q^6+q^5+q^4+q^3+q^2+q+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^4-q^3+q^2-q+1)$ $(q^4+q^3+q^2+q+1)(q^2+q+1)^2(q^2-q+1)^2(q^4+1)^2(q-1)^5(q+1)^5(q^4-q^2+1)^5/q^{43}$
$x_3 \otimes x_1 \otimes x_1$	$-(q^2+1)^4(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5$ $+q^4+q^3+q^2+q+1)(q^{12}-q^{10}+q^8+q^6+q^4-q^2+1)(q^2+q+1)^2(q^2-q+1)^2(q^4+1)^2$ $(q-1)^4(q+1)^4(q^4-q^2+1)^5/q^{42}$

Figure 36: Nonzero coefficients of x in the lattice $L(\mathbb{C}_q^{3,H}(X))$

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