Geometric Complexity Theory VII: Nonstandard quantum group for the plethysm problem

(extended abstract)

Dedicated to Sri Ramakrishna

Ketan D. Mulmuley *
The University of Chicago

http://ramakrishnadas.cs.uchicago.edu

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Abstract

This article describes a *nonstandard* quantum group that may be used to derive a positive formula for the plethysm problem, just as the standard (Drinfeld-Jimbo) quantum group can be used to derive the positive Littlewood-Richardson rule for arbitrary complex semisimple Lie groups. The sequel [GCT8] gives conjecturally correct algorithms to construct canonical bases of the coordinate rings of these nonstandard quantum groups and canonical bases of the dually paired nonstandard deformations of the symmetric group algebra. A positive $\#P$-formula for the plethysm constant follows from the conjectural properties of these canonical bases and the duality and reciprocity conjectures herein.

1 Introduction

The following is a fundamental problem in representation theory [GCT6, Mc, St]:

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Problem 1.1 (Plethysm problem)

Find an explicit positive (#P-) formula in the spirit of the Littlewood-Richardson rule for the plethysm constant $a^{\pi}_{\lambda,\mu}$*. For given partitions* λ, μ *and* π*, this is the multiplicity of the irreducible representation* $V_\pi(H)$ *of* $H = GL_n(\mathbb{C})$ *in the irreducible representation* $V_\lambda(G)$ *of* $G = GL(X)$*, where* $X = V_\mu = V_\mu(H)$ *is an irreducible representation of* H*. Here* $V_\lambda(G)$ *is considered an* H*-module via the representation map* $\rho : H \to G$*.*

(Generalized plethysm problem):

The same as above, letting H *be any complex, semisimple (or, more generally, reductive) classical Lie group,* λ *a dominant weight of* G*,* π *and* μ *dominant weights of* H*.*

This article describes a quantum group that may be used to derive such a positive formula, just as the standard (Drinfeld-Jimbo) quantum group [Dri, Ji, RTF] can be used to derive the positive Littlewood-Richardson rule for arbitrary complex semisimple Lie groups [Kas1, Li, Lu2]; the results here were announced in [GCT4] (most of the results here also hold for nonclassical H, though we shall only worry about classical H here). For the significance of a positive formula in the context of geometric complexity theory, see [GCTflip1]. The approach that we wish to follow is:

1. Find a quantization of the homomorphism
$$H \to G \tag{1}$$
of the form
$$H_q \to G_q^H, \tag{2}$$
where H_q is the standard Drinfeld-Jimbo quantization of H, and G_q^H is the new nonstandard quantization of G that we seek.

2. Develop a theory of canonical (local/global crystal) bases for the representations of G_q^H in the spirit of the canonical bases [Kas1, Lu1] for the representations of the standard quantum group.

3. Derive the required explicit positive formula for the plethysm constant from the properties of the canonical bases.

The following addresses the first step.

Theorem 1.2 *(cf. Section 2) There exists a possibly singular quantum group G_q^H such that the homomorphism (1) can be quantized in the form (2).*

Furthermore, all finite dimensional polynomial representations of G_q^H are completely reducible, and a quantum analogue of the Peter-Weyl theorem holds for the matrix coordinate ring of G_q^H.

For the precise meaning of the various terms here, see Section 2. Here and in what follows, we assume that the base field is $\mathbb{C} = \mathbb{C}(q)$, q complex. But a suitable algebraic extension of $Q(q)$ will also suffice for our purposes; see Section 6 for a discussion on the base field.

When $H = G$, G_q^H specializes to the standard quantum group H_q. When $H = GL(V) \times GL(W)$, $G = GL(X)$, $X = V \otimes W$ with natural H-action, it reduces to the quantum group in [GCT4] for the Kronecker problem.

We call G_q^H the *nonstandard quantum group* associated with the embedding (1). It can be singular in general. That is, its determinant may vanish, and hence, the antipode need not exist. Strictly speaking, it should hence be called a nonstandard quantum semi-group. We still use the term group, because this object has characteristic features of the standard quantum group, such as semisimplicity of polynomial representations, Peter-Weyl theorem, and most importantly, conjectural existence of canonical bases for its representations and the matrix coordinate ring.

We also construct (Section 5) a *nonstandard quantization* $\mathcal{B}_r^H = \mathcal{B}_r^H(q)$ of the group algebra $\mathbb{C}[S_r]$ of the symmetric group S_r whose relationship with G_q^H is conjecturally akin to that of the Hecke algebra with the standard quantum group. Specifically, let X_q denote the irreducible representation $V_{q,\mu}$ of H_q with highest weight μ; it is the usual quantization of $X = V_\mu$. Then:

Conjecture 1.3 (Nonstandard duality)

(1) The left action of G_q^H on $X_q^{\otimes r}$ and the right action $\mathcal{B}_r^H(q)$ on $X_q^{\otimes r}$ determine each other.

(2) There is a one-to-one correspondence between the irreducible polynomial representations of G_q^H of degree r and the irreducible representations of $\mathcal{B}_r^H$ so that, as a bimodule,

$$X_q^{\otimes r} = \bigoplus_\alpha W_{q,\alpha} \otimes T_{q,\alpha}, \tag{3}$$

where $W_{q,\alpha}$ runs over the irreducible polynomial representations of G_q^H of degree r, and $T_{q,\alpha}$ denotes the irreducible representation of $\mathcal{B}_r^H(q)$ in correspondence with $W_{q,\alpha}$.

The irreducible representations $W_{q,\alpha}$ here need not be q-deformations of the irreducible representations of G, because G_q^H is, in general, a nonflat deformation of G. This means the Poincare series of G_q^H need not coincide with that of G. Our first goal is to associate with each Weyl module V_λ of G a possibly reducible representation $V_{q,\lambda}^H$ of G_q^H, called the *q-analogue* of V_λ, so that

$$\lim_{q\to 1} V_{q,\lambda}^H \cong V_\lambda$$

as an H-module. In this context:

Conjecture 1.4 (Nonstandard reciprocity) *Let λ be a partition of size r. Let*

$$V_{q,\lambda}^H = \bigoplus_\alpha m_\lambda^\alpha W_{q,\alpha},$$

where m_λ^α denotes the multiplicity of the Specht module S_λ of the symmetric group S_r in $T_{q,\alpha}(1) = lim_{q\to 1} T_{q,\alpha}$, as defined in Section 6. Then $V_{q,\lambda}^H$ is a q-analogue of V_λ in the sense defined above.

Thus the multiplicity of the G_q^H-module $W_{q,\alpha}$ in $V_{q,\lambda}^H$ is equal to the multiplicity of the Specht module S_λ in the specialization of $T_{q,\alpha}$ at $q = 1$.

A more refined form of this conjecture is given in Section 6. Both duality and reciprocity are supported by experimental evidence; cf. Section 7.

By the conjectural reciprocity,

$$a_{\lambda,\mu}^\pi = \sum_\alpha m_\lambda^\alpha n_\pi^\alpha,$$

where n_π^α is the multiplicity of the irreducible H_q-module $V_{q,\pi}$ in $W_{q,\alpha}$. Hence Problem 1.1 can be decomposed into the following two subproblems:

(P1): Find a positive ($\#P$-) formula for the multiplicity n_π^α.

(P2): Find a positive ($\#P$-) formula for the multiplicity m_λ^α.

The article [GCT8] gives conjecturally correct algorithms to construct a canonical basis of the matrix coordinate ring of G_q^H whose conjectural properties would imply a positive formula as needed in the first problem, and a canonical basis of $\mathcal{B}_r^H$ whose conjectural properties would imply a positive formula as needed in the second problem.

At present, we cannot prove correctness of these algorithms nor the required conjectural properties, because we are unable to deal with the high complexity of the nonstandard quantum group. Specifically, as we shall see in Section 4, the formulae for the minors of the nonstandard group turn out to be highly nonelementary in contrast to the elementary formulae for the minors of the standard quantum group. The coefficients of these formulae depend on the multiplicative structural constants of canonical bases akin to the canonical basis of the coordinate ring of the standard quantum group constructed by Kashiwara and Lusztig [Kas2, Lu2]. To get explicit formulae for these structural constants, one needs interpretations for them akin to the interpretations for the Kazhdan-Lusztig polynomials and multiplicative structural constants of the canonical basis of the coordinate ring of the standard quantum group in terms of perverse sheaves [KL2, Lu1, BBD]. Thus, the linear algebra for the nonstandard quantum group–i.e. the theory of its minors–is already highly nonelementary in contrast to the linear algebra for the standard quantum group. This is why its representation theory may turn out to be far more complex. In particular, we cannot explicitly construct nor classify its irreducible polynomial representations. Of course, all this and much more would follow if correctness of the algorithms in [GCT8] for constructing canonical bases and their conjectural properties can be proved.

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2 Nonstandard quantum group

We describe in this section the construction of the nonstandard quantum group G_q^H in Theorem 1.2. The reader may refer to [GCT4] for the full details in a nontrivial special case of the plethysm problem, called the Kronecker problem. For the sake of simplicity, we assume here that $H = GL(V)$ (type A). Let $X = V_\mu(H)$ be its irreducible polynomial representation. The goal is to quantize the homomorphism

$$H = GL(V) \to G = GL(X).$$

Let $\mathcal{H}$ and $\mathcal{G}$ be the Lie algebras of H and G. We follow the terminology in [Kli], which will be our standard reference on quantum groups.

The standard quantum group $H_q = GL_q(V)$ associated with $GL(V)$ can be defined by first constructing the coordinate algebra $\mathcal{O}(M_q(V))$ of the standard quantum matrix space $M_q(V)$ as a suitable FRT-algebra [RTF]. The coordinate ring $\mathcal{O}(GL_q(V))$ of $GL_q(V)$ is obtained by localizing $\mathcal{O}(M_q(V))$ at the suitably defined quantum determinant. The Drinfeld-Jimbo universal enveloping algebra $U_q(\mathcal{G})$ [Dri, Ji] of $GL_q(V)$ can then be defined dually. Specifically, let J be the maximal ideal of the elements in $\mathcal{O}(M_q(V))$ which vanish at the identity–i.e. on which ϵ, the counit, vanishes. Then $U_q(\mathcal{G})$ can be identified with the space of linear functions on $\mathcal{O}(M_q(V))$ which vanish on J^r for some integer $r > 0$ depending on the linear function.

Analogously, we first construct the nonstandard matrix coordinate ring $\mathcal{O}(M_q^H(X))$ of the (virtual) nonstandard matrix space $M_q^H(X)$, and then define the nonstandard quantized universal enveloping algebra $U_q^H(\mathcal{G})$ by dualization. We define the nonstandard quantum group G_q^H as the virtual object whose universal enveloping algebra is $U_q^H(\mathcal{G})$. The construction would yield natural bialgebra homomorphisms from $U_q(\mathcal{H})$ to $U_q^H(\mathcal{G})$ and from $\mathcal{O}(M_q^H(X))$ to $\mathcal{O}(M_q(V)$, thereby giving the desired quantizations of the homomorphisms $U(\mathcal{H}) \to U(\mathcal{G})$ and $\mathcal{O}(M(X)) \to \mathcal{O}(M(V))$. This is what is meant by the quantization (2) of the map (1). The determinant of G_q^H may vanish, and hence, we cannot, in general, define its coordinate ring $\mathcal{O}(G_q^H)$ by localizing $\mathcal{O}(M_q^H(X))$. Fortunately, this will not matter since the coordinate ring $\mathcal{O}(M_q^H(X))$ and the nonstandard quantized algebra $\mathcal{B}_r^H(q)$ (Section 5) together contain conjecturally all the information that we need (cf. Conjecture 1.4), and have properties similar to that of the standard matrix coordinate ring $\mathcal{O}(M_q(V))$ and the Hecke algebra; cf. Theorem 2.1 below.

The *nonstandard matrix coordinate ring* $\mathcal{O}(M_q^H(X))$ is constructed as follows. Let $\hat{R}^H_{X,X}$ be the $\hat{R}$ matrix of $X_q = V_{q,\mu}$ considered as an H_q-module [Kli]. Here and in what follows, we sometimes denote X_q by X; the meaning should be clear from the context. It is well known that $\hat{R}^H_{X,X}$ is diagonalizable and that its each eigenvalue is of the form $+$ or $-q^{a/2}$ for some integer a [Kli]. Let

$$I = P^{+,H}_{X,X} + P^{-,H}_{X,X}, \tag{4}$$

be the associated spectral decomposition of the identity, where $P^{+,H}_{X,X}$ and $P^{-,H}_{X,X}$ denote the projections of $X_q \otimes X_q$ on the eigenspaces of $\hat{R}^H_{X,X}$ for eigenvalues with $+$ and $-$ sign, respectively. Let $\mathbf{u}$ be a variable matrix specifying a generic transformation from X to X. Let u^i_j denote its variable entries. Then $\mathcal{O}(M_q^H(X))$ is defined to be the FRT bialgebra [RTF] associated with the transformation $P^{+,H}_{X,X}$, or equivalently, $P^{-,H}_{X,X}$. That is, it is the quotient of $\mathbb{C}\langle u^i_j\rangle$ modulo the relations

$$P^{+,H}_{X,X}(\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\otimes\mathbf{u})P^{+,H}_{X,X}, \tag{5}$$

or equivalently,

$$P^{-,H}_{X,X}(\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\otimes\mathbf{u})P^{-,H}_{X,X}. \tag{6}$$

An alternative definition of $\mathcal{O}(M^H_q(X))$ is as follows. Let $S^H_q(X\otimes X)$, the *symmetric subspace* of $X\otimes X$, be the image of $P^{+,H}_{X,X}$, and $A^H_q(X\otimes X)$, the *antisymmetric subspace* of $X\otimes X$, the image of $P^{-,H}_{X,X}$ [Kli]. In other words, $S^H_q(X\otimes X)$ is defined by the equation

$$P^{-,H}_{X,X}\mathbf{x_1}\mathbf{x_2} = 0, \tag{7}$$

where $\mathbf{x_1} = \mathbf{x}\otimes\mathbf{I}$ and $\mathbf{x_2} = \mathbf{I}\otimes\mathbf{x}$, and $A^H_q(X\otimes X)$ is defined by the equation

$$P^{+,H}_{X,X}\mathbf{x_1}\mathbf{x_2} = 0. \tag{8}$$

The *braided symmetric algebra* [BZ, Ro] $\mathbb{C}^H_q[X]$ of X is defined to be the algebra over the entries x_i's of $\mathbf{x}$ subject to the relation (7). It will be called the coordinate ring of the virtual *quantum space* X^H_{sym}. Similarly, the *braided exterior algebra* $\wedge^H_q[X]$ of X is defined to be the algebra over the entries x_i's of $\mathbf{x}$ subject to the relation (8). It will called the coordinate ring of the virtual *quantum space* $X^H_\wedge$. Let $\mathbb{C}^{H,r}_q[X]$ and $\wedge^{H,r}_q[X]$ be the degree r components of $\mathbb{C}^H_q[X]$ and $\wedge^H_q[X]$, respectively. It is known [BZ] that the dimensions of $\mathbb{C}^{H,r}_q[X]$ and $\wedge^{H,R}_q$ are bounded by the dimensions of the classical $\mathbb{C}^r[X]$ and $\wedge^r[X]$, respectively. But unlike in the standard setting, the dimensions can be strictly less [BZ, Ro]. That is, $\mathbb{C}^H_q[X]$ and $\wedge^H_q[X]$ are, in general, nonflat deformations of the classical symmetric and exterior algebras $\mathbb{C}[X]$ and $\wedge[X]$. For example, $\wedge^{H,3}_q[X] = 0$ when $\mathcal{H} = sl_2(\mathbb{C})$ and X is the four dimensional irreducible representation of $sl_2(\mathbb{C})$ [BZ].

The equation (5) or (6) after reformulation just says that the defining relation (7) of X^H_{sym}–or equivalently, the defining relation (8) of $X^H_\wedge$–is preserved under the left and right actions of $\mathbf{u}$ on $\mathbf{x}$ given by $\mathbf{x}\to\mathbf{u}\mathbf{x}$ and $\mathbf{x}^t\to\mathbf{x}^t\mathbf{u}$.

This means $\mathbb{C}^H_q[X]$ and $\wedge^H_q[X]$ have left and right coactions of $\mathcal{O}(M^H_q(X))$. We define the left and right *nonstandard minors* of G^H_q to be the matrix coefficients (in a suitable basis specified later) of the left and right coactions on $\wedge_q^H[X]$. If $\wedge_q^{H,\dim(X)}[X] \neq 0$, then we define the determinant of G_q^H to be the matrix coefficient of the action of $\mathcal{O}(M_q^H(X))$ on $\wedge_q^{H,\dim(X)}[X]$. But it can vanish, as it does for $\mathcal{H} = sl_2(\mathbb{C})$, $\dim(X) = 4$. The nonstandard minors will be discussed in more detail in Section 4.

Let J be the ideal of elements in $\mathcal{O}(M_q^H(X))$ on which the counit ϵ vanishes. Then the *nonstandard universal enveloping algebra* $U_q^H(\mathcal{G})$ is defined to be the space of linear functions of $\mathcal{O}(M_q^H(X))$ which vanish on J^r for some $r > 0$ depending on the linear function.

The following is a precise form of Theorem 1.2.

Theorem 2.1 *(1) There is a natural bialgebra homomorphism from $\mathcal{O}(M_q^H(X))$ to $\mathcal{O}(M_q(V))$. This gives the desired quantization of the homomorphism $\mathcal{O}(M(X)) \to \mathcal{O}(M(V))$.*

(2) The matrix coordinate ring $\mathcal{O}(M_q^H(X))$ of G_q^H is cosemisimple. Hence, its every finite dimensional corepresentation is completely reducible as a direct sum of irreducible corepresentations.

(3) The q-analogue of the Peter-Weyl theorem holds: i.e.,

$$\mathcal{O}(M_q^H(X)) = \bigoplus_\alpha W_{q,\alpha}^* \otimes W_{q,\alpha},$$

where $W_{q,\alpha}$ runs over all irreducible corepresentations of $\mathcal{O}(M_q^H(X))$.

(4) The nonstandard enveloping algebra $U_q^H(\mathcal{G})$ is a bialgebra with a compact real form (a $*$-structure) such that $X_q^{\otimes r}$ is its unitary representation with respect to the Hermitian form on $X_q^{\otimes r}$ induced by the standard Hermitian form on X_q. There is a bialgebra homomorphism form $U_q(\mathcal{H})$ to $U_q^H(\mathcal{G})$. This gives a desired quantization of the homomorphism $U(\mathcal{H}) \to U(\mathcal{G})$.

Here the standard Hermitian form on X_q is the one that is U_q-invariant, where $U_q \subseteq H_q$ is the compact real form (the unitary subgroup) of H_q in the sense of Woronowicz [W]. The special case of this theorem in the context of the Kronecker problem was proved in [GCT4] on the basis of Woronowicz's work [W]. The latter is no longer applicable in the general context here, since the determinant of G_q^H may vanish, and hence, we cannot, in general, convert $\mathcal{O}(M_q^H(X))$ into a Hopf algebra by localization at the determinant. Fortunately, this does not matter since $U_q^H(\mathcal{G})$ still has a compact real form, whose existence can be proved using the spectral properties of $\hat{R}_{X,X}^H$.

We also call $W_{q,\alpha}$ here a *polynomial representation* of G_q^H. By a polynomial representation of $U_q^H(\mathcal{G})$ we mean a representation that is induced by a (finite dimensional) corepresentation of $\mathcal{O}(M_q^H(X))$. It is completely reducible by cosemsimplicity of $\mathcal{O}(M_q^H(X))$. It may be conjectured that every finite dimensional representation of $U_q^H(\mathcal{G})$ is completely reducible (as in the standard case), though we shall not need this more general fact.

The standard Drinfeld-Jimbo enveloping algebra has an explicit presentation in the form of explicit generators (e_i, f_i, K_i) and explicit relations among them. It will be interesting to find an analogous explicit presentation for $U_q^H(\mathcal{G})$; cf. Section 4 for the problems that arise in this context.

3 Nonstandard q-Schur algebra

In the standard setting, the q-Schur algebra $\mathcal{A}_r = \mathcal{A}_r(q)$ is defined to be the dual $\mathcal{O}(M_q(V))^*r$ of the degree r component $\mathcal{O}(M_q(V))_r$ of the standard matrix coordinate algebra $\mathcal{O}(M_q(V))$. Thus $\mathcal{A}_r(q)$ acts on $V^{\otimes r}$ from the left. It is known [Kli] that it is the centralizer in $\mathrm{End}(V^{\otimes r})$ of the right action of the Hecke algebra $\mathcal{H}_r(q)$ on $V^{\otimes r}$.

Analogously, we define the *nonstandard q-Schur algebra* $\mathcal{A}_r^H = \mathcal{A}_r^H(q)$ to be the dual $\mathcal{O}(M_q^H(X))^*r$ of the degree r component $\mathcal{O}(M_q^H(X))_r$ of the nonstandard matrix coordinate algebra $\mathcal{O}(M_q^H(X))$. Thus $\mathcal{A}_r^H(q)$ acts on $X^{\otimes r}$ from the left. As per the nonstadard duality conjecture (Conjecture 1.3), it is the centralizer in $\mathrm{End}(X^{\otimes r})$ of the right action of the nonstandard quantized algebra $\mathcal{B}_r^H(q)$ (cf. Section 5) on $X^{\otimes r}$.

Every irreducible corepresentation $W_{q,\alpha}$ of $\mathcal{O}(M_q^H(X))$ of degree r can also be considered as a representation of $\mathcal{A}_r^H(q)$, and conversely, every irreducible representation of $\mathcal{A}_r^H(q)$ arises in this way. Theorem 2.1 now immediately implies:

Theorem 3.1 *(1) The nonstandard q-Schur algebra $\mathcal{A}_r^H(q)$ is semisimple. Hence, its every finite dimensional representation is completely reducible as a direct sum of irreducible representations.*

(2) The q-analogue of the Peter-Weyl theorem in this case is the Wederburn structure theorem for $\mathcal{A}_r^H(q)$:

$$\mathcal{A}_r^H(q) = \bigoplus_{\alpha} W_{q,\alpha}^* \otimes W_{q,\alpha},$$

where $W_{q,\alpha}$ runs over all irreducible representations of $\mathcal{A}_r^H(q)$.

(3) The nonstandard q-Schur algebra $\mathcal{A}_r^H(q)$ has a compact real form (a $*$-structure) such that $X_q^{\otimes r}$ is its unitary representation with respect to the Hermitian form on $X_q^{\otimes r}$ induced by the standard Hermitian form on X_q.

4 Nonstandard minors

In this section, we give a conjectural formula for the Laplace expansion of the minors of G_q^H. The Laplace expansion for the standard quantum group $GL_q(V)$ is based on the simple relation defining the standard exterior algebra $\wedge_q[V]$, namely

$$v_i^2 = 0 \quad \text{and} \quad v_i v_j = -q^{-1} v_j v_i, \quad \text{for} \quad i < j.$$

This explains why the Laplace expansion in the standard setting is obtained from the classical Laplace expansion by simply substituting $-q$ for -1. We need a similar explicit formula for multiplication in $\mathbb{C}_q^H[X]$ to get an explicit formula for Laplace expansion in the nonstandard setting.

4.1 Kronecker problem

We begin with a special case that arises in the context of the Kronecker problem [GCT4] when $H = GL(V) \times GL(W)$ and $X = V \otimes W$, with the natural H-action. The article [GCT4] gives a formula for the column or row expansion of the minor of $GL_q^H(X)$ in this special case in terms of fundamental Clebsch-Gordon coefficients for the standard quantum groups $GL_q(V)$ and $GL_q(W)$. But this formula cannot be extended for the general Laplace expansion since Clebsch-Gordon coefficients are not well defined when the underlying tensor products do not have multiplicity-free decompositions as in the fundamental case. Here we give a formula for general Laplace expansion of the minors of $GL_q^H(X)$ in this case.

We begin by recalling that when $V = W^*$ the braided symmetric algebra $\mathbb{C}_q^H[X] = \mathbb{C}^H[W^* \otimes W]$ is isomorphic to the matrix coordinate ring $\mathcal{O}(M_q(W))$ of the standard matrix space $M_q(W)$ [GCT4]. For this, we have:

Theorem 4.1 *(Kashiwara and Lusztig [Kas2, Lu2]) The coordinate ring $\mathcal{O}(M_q(W))$ has an (upper) canonical basis.*

This can be naturally and easily extended to:

Theorem 4.2 *The braided symmetric coordinate algebra* $\mathbb{C}_q^H[X] = \mathbb{C}_q^H[V \otimes W]$*,* $H = GL(V) \times GL(W)$*, has an (upper) canonical basis.*

The exterior form of this result is:

Theorem 4.3 *The exterior coordinate algebra* $\wedge_q^H[V \otimes W]$*,* $H = GL(V) \times GL(W)$*, also has an (upper) canonical basis.*

Lusztig [Lu2] has conjectured that the multiplicative and comultiplicative structural constants of the canonical basis of $\mathcal{O}(M_q(W))$ are polynomials in q and q^{-1} with nonnegative integer coefficients; i.e., belong to $\mathbb{N}[q, q^{-1}]$. Analogous conjecture can be made for $\wedge_q^H[V \otimes W]$. Specifically, it can be conjectured that for any canonical basis elements b and b' in $\wedge_q^H[V \otimes W]$:

$$bb' = \sum_{b''} \epsilon(b, b', b'') c_{b,b'}^{b''} b'', \tag{9}$$

where the sign $\epsilon(b, b', b'')$ is 1 or -1 and the coefficient $c_{b,b'}^{b''} \in \mathbb{N}[q, q^{-1}]$. And conversely, any $b'' \in \wedge_q^{H,r''}[V \otimes W]$ can be expressed as:

$$b'' = \sum_{b,b'} \epsilon'(b, b', b'') d_{b''}^{b,b'} bb', \tag{10}$$

where b and b' run over elements of $\wedge_q^{H,r}[V \times W]$ and $\wedge_q^{H,r'}[V \times W]$ respectively with $r'' = r + r'$, the sign $\epsilon'(b, b', b'')$ is 1 or -1, and $d_{b''}^{b,b'} \in \mathbb{N}[q, q^{-1}]$. To prove nonnegativity of the coefficients of $c_{b,b'}^{b''}$ and $d_{b''}^{b,b'}$, one needs interpretations for them in terms of perverse sheaves [BBD] in the spirit of Kazhdan-Lusztig [KL2] and Lusztig [Lu1].

We now define the (left or right) minors of G_q^H with respect to the canonical basis $\wedge_q^{H,r}[V \otimes W]$ to be the matrix coefficients of the (left or right) coaction of $\mathcal{O}(M_q^H(V \otimes W))$. We shall call them (left or right) *canonical minors*. Then:

Theorem 4.4 *A canonical minor of degree* r'' *of* $GL_q^H(X)$*,* $H = GL(V) \times GL(W)$*, admits a Laplace expansion in terms of canonical minors of degree* r *and* r' *with* $r'' = r + r'$*. The coefficients of this Laplace expansion are quadratic forms in the structural constants* $c_{b,b'}^{b''}$ *and* $d_{b''}^{b,b'}$ *above.*

An explicit formula for Laplace expansion here (omitted) is similar to the one in Proposition 6.1 of [GCT4] with these structural constants in place of the Clebsch-Gordon coefficients there (which are not well defined for general Laplace expansion).

4.2 General nonstandard setting

Now let us turn to the general case. The conjecturally correct algorithm in [GCT8] for constructing a canonical basis of $\mathcal{O}(M_q^H(X))$ also yields, as a byproduct, conjectural canonical bases of $\wedge_q^H[X]$ and $\mathbb{C}_q^H[X]$ as implicitly sought in [BZ]. We define the (left or right) minors of G_q^H in general to be the matrix coefficients of the (left or right) coaction of $\mathcal{O}(M_q^H(X))$ in this canonical basis of $\wedge_q^H[X]$. We call these *nonstandard canonical minors*, or simply *nonstandard minors*.

One can define structural constants $c_{b,b'}^{b''}$ and $d_{b''}^{b,b'}$ analogous to the ones in (9) and (10) in this case. With this:

Theorem 4.5 *Analogue of Theorem 4.4 holds in general.*

Laplace expansion in the standard setting is used as a straightening relation to construct standard monomial bases of the coordinate ring and irreducible representations of $GL_q(X)$. In this sense, Laplace expansion is a mother relation that governs the representation theory of the standard quantum group. Similarly, the nonstandard Laplace expansions in Theorems 4.4 and 4.5 are expected to be mother relations governing the representation theory of the nonstandard quantum group G_q^H. In particular, an explicit interpretation for the structural coefficients $c_{b,b'}^{b''}$ and $d_{b''}^{b,b'}$ akin to the ones based on perverse sheaves for the Kazhdan-Lusztig polynomials [KL2] and the multiplicative structural constants of the canonical basis for the standard quantum group [Lu2] is necessary to get fully explicit formulae for the nonstandard minors, and hence, for constructing explicit bases for the irreducible polynomial representations and the matrix coordinate ring of G_q^H. In particular, this seems necessary for proving correctness of the algorithms in [GCT8] for constructing nonstandard canonical bases for the polynomial representations and the matrix coordinate ring of G_q^H. This also seems necessary for finding an explicit presentation of the nonstandard universal enveloping algebra $U_q^H(\mathcal{G})$ in the spirit of the explicit presentation of the Drinfeld-Jimbo enveloping algebra. Specifically, we expect the coefficients occuring in such an explicit presentation to depend on the structural constants such as $c_{b,b'}^{b''}$ and $d_{b''}^{b,b'}$ above.

5 Nonstandard quantized algebra

We now construct a nonstandard quantization $B_r^H(q)$ of the symmetric group ring $\mathbb{C}[S_r]$ which conjecturally has the same relationship with G_q^H that the Hecke algebra $\mathcal{H}_r(q)$, the standard deformation of $\mathbb{C}[S_r]$, has with the standard quantum group. For the sake of simplicity, we assume that $H = GL(V)$.

Choose a standard embedding of $X = V_\mu(H)$ in $V^{\otimes d}$, where d is the size of the partition μ. That is, choose a Young symmetrizer $c_\mu \in \mathbb{C}[S_r]$ such that $V^{\otimes d} \cdot c_\mu$, the image of $V^{\otimes d}$ under the right action of c_μ, is isomorphic to $X = V_\mu(H)$. Let $z_\mu \in \mathcal{H}_d(q)$ be the quantization of c_μ such that $V_q^{\otimes d} \cdot z_\mu \cong X_q = V_{q,\mu}$. Here V_q denotes the quantization of V and $V_{q,\mu}$ the irreducible H_q module with highest weight μ. An explicit expression of z_μ may be found in [DJ]. Let $Z_q = V_q^{\otimes d}$. Let $\hat{R}_{Z,Z}^H$ denote the $\hat{R}$-matrix of Z_q as an H_q-module. Let $r_Z \in \mathcal{H}_{2d}(q)$, $1 \leq i < r$, be the element whose right action on $Z_q \otimes Z_q = V_q^{\otimes 2d}$ coincides with the action of $\hat{R}_{Z,Z}^H$. One can easily write down an explicit expression for r_Z in terms of the generators of $\mathcal{H}_{2d}(q)$.

Now consider the right action of $\mathcal{H}_s(q)$, $s = dr$, on $Z_q^{\otimes r} = V_q^{\otimes s}$, which commutes with the left action of $H_q = GL_q(V)$. Let $r_{Z,i} \in \mathcal{H}_s(q)$, $1 \leq i < r$, be the element whose right action on $Z_q^{\otimes r}$ coincides with the action of $\hat{R}_{Z,Z}^H$ on the product of the i-th and $(i+1)$-st factors of $Z_q^{\otimes r}$. Thus $r_{Z,i}$ is the image of r_Z under the obvious embedding of $\mathcal{H}_{2d}(q)$ in $\mathcal{H}_s(q)$ depending on i. One can thus write down an explicit expression for $r_{Z,i}$ in terms of the generators of $\mathcal{H}_s(q)$. Let

$$r_{X,i}^H = z_{\lambda,i} \cdot z_{\lambda,i+1} \cdot r_{Z,i},$$

where $z_{\lambda,i} \in \mathcal{H}_s(q)$ denotes an explicit element whose action on the i-th factor of $Z_q^{\otimes r}$ coincides with the action of z_λ on that factor–it is the image of z_λ under the obvious embedding of $\mathcal{H}_d(q)$ in $\mathcal{H}_s(q)$ depending on i. Then the right action of $r_{X,i}^H$ on $Z_q^{\otimes r}$ corresponds to the action of $\hat{R}_{X,X}^H$ on the product of the i-th and $(i+1)$-st factors of $X_q^{\otimes d} \subseteq Z_q^{\otimes d}$. Let $p_{X,i}^{+,H}, p_{X,i}^{-,H} \in H_s(q)$ be the polynomials in $r_{X,i}^H$ whose actions on $Z_q^{\otimes r}$ correspond to the actions of the positive and negative projection operators $P_{X,X}^{+,H}$ and $P_{X,X}^{-,H}$ in eq. (4) on the tensor product of the i-th and $(i+1)$-st factors of $X_q^{\otimes d} \subseteq Z_q^{\otimes d}$; one can write down these polynomials explicitly, using the known explicit spectral form of $\hat{r}^H_{X,i}$.

We define the *nonstandard quantized algebra* $\mathcal{B}^H_r(q)$ to be the subalgebra of $\mathcal{H}_s(q)$ generated by the explicit elements $p^{+,H}_{X,i}$, or equivalently, $p^{-,H}_{X,i}$. In general, it is a nonflat deformation of $\mathbb{C}[S_r]$. That is, its dimension can be larger than that of $\mathbb{C}[S_r]$. It can be shown to be semisimple. Its right action on $X_q^{\otimes r}$ commutes with the left action G^H_q by the defining equation (5) of G^H_q. Conjecture 1.3 says that its relationship with G^H_q is akin to that of $\mathcal{H}_r(q)$ with the standard quantum group $G_q = GL_q(X)$.

The Hecke algebra has an explicit presentation in terms of explicit relations among its generators. It will be interesting to find an analogous explicit presentation for $\mathcal{B}^H_r(q)$. Its complexity would be much higher than that of the Hecke algebra as indicated by the concrete computations in [GCT4]. Specifically, we expect an explicit presentation for $\mathcal{B}^H_r(q)$ with defining relations whose coefficients are akin to the structural constants $c^{b''}_{b,b'}$ and $d^{b,b'}_{b''}$ in Section 4 and have a topological interpretation akin to the one for Kazhdan-Lusztig polynomials. Such an explicit presentation is needed to prove correctness of the algorithm in [GCT8] to construct a canonical basis of $\mathcal{B}^H_r$.

Remark: We can also define a (possibly singular) quantum group $\hat{G}^H_q$, instead of G^H_q, by substituting $\hat{R}^H_{X,X}$ in place of $P^{+,H}_{X,X}$ in the defining equation (5). One can then define a deformation $\hat{\mathcal{B}}^H_q(r)$ of $\mathbb{C}[S_r]$ that is conjecturally paired with $\hat{G}^H_q$, as G^H_q is with $\mathcal{B}^H_q(r)$. The main results (semisimplicity, and q-analogue of the Peter-Weyl theorem) also hold for these objects. Furthermore, variants of the algorithms in [GCT8] can be conjectured to provide canonical bases for these as well. However, the Poincare series of $\hat{G}^H_q$ is much smaller than that of G^H_q, and for this and other reasons, it does not seem possible to use these objects in the context of the plethysm problem. However, these may be interesting intermediate quantum objects to study nevertheless.

6 Refined reciprocity

We now describe a refinement of the reciprocity conjecture (Conjecture 1.4) that specifies precisely how the decomposion (3) of $X_q^{\otimes r}$,

$$X_q^{\otimes r} = \bigoplus W_{q,\alpha} \otimes T_{q,\alpha}, \tag{11}$$

as a $G_q^H \times \mathcal{B}_r^H(q)$-bimodule, tends to the decomposition

$$X^{\otimes r} = \bigoplus_\lambda V_\lambda \otimes S_\lambda \tag{12}$$

of $X^{\otimes r}$ as a $G \times S_r$-bimodule, as $q \to 1$, and gives an explicit realization within $X_q^{\otimes r}$ of the q-analogue $V_{q,\lambda}^H$ of V_λ as in Conjecture 1.4. Here, as usual, V_λ denotes the Weyl module of G, and S_λ the Specht module of S_r.

First, we have to define the multiplicity m_λ^α of a Specht module S_λ in the specialization $T_{q,\alpha}(1)$ of $T_{q,\alpha}$ at $q = 1$. In this context, it may be remarked that though $\mathcal{B} = \mathcal{B}_r^H(q)$ is semisimple, its specialization $\mathcal{B}(1)$ at $q = 1$ need not be semisimple; see Section 7.1 for an example. Clearly, every representation of S_r is also a representation of $\mathcal{B}(1)$, though not always conversely. But it may be conjectured that every irreducible $\mathcal{B}(1)$-representation is also an irreducible S_r-representation, i.e., a Specht module. Fix any (maximal) composition series of $T_{q,\alpha}(1)$ as a $\mathcal{B}(1)$-module. We define the multiplicity m_λ^α to be the number of factors in this (or any such) series that are isomorphic to the specht module S_λ.

Since $\mathcal{B}$ is semisimple (cf. Section 5), it admits a Wederburn structure decomposition of the form

$$\mathcal{B} = \bigoplus U^\alpha, \quad U^\alpha = T_{q,\alpha,L} \otimes T_{q,\alpha,R}, \tag{13}$$

where α is as in (11), and $T_{q,\alpha,L}$ and $T_{q,\alpha,R}$ denote the left and right irreducible $\mathcal{B}$-modules indexed by α. We call this a *complete* Wederburn structure decomposition. Here we are assuming that the base field is $\mathbb{C} = \mathbb{C}(q)$, q complex. This complete decomposition would also hold if the base field is instead an appropriate algebraic extension K of $Q(q)$. In the standard setting of Hecke algebras, $K = Q(q)$ suffices. This need not be so in the nonstandard setting. That is, an algebraic extension of $Q(q)$ may be actually necessary for a complete decomposition of the above form to hold; see Section 7.1 for an example. If the base field is $Q(q)$, each U^α in the Wederburn structure decomposition need not be, in general, of the form $T_{q,\alpha,L} \otimes T_{q,\alpha,R}$ as above, but rather it would be isomorphic to the endomorphism ring of $T_{q,\alpha}$ over the division algebra $End_{\mathcal{B}}(T_{q,\alpha})$. One has to take similar variations of the nonstandard q-analogue of the Peter-Weyl theorem (Theorem 2.1 (3)) and the duality conjecture (Conjecture 1.3) if the base field is $Q(q)$. However, for the reciprocity conjecture, it is necessary to take the base field as $\mathbb{C}(q)$, q complex, or an algebraic extension K of $Q(q)$ as described above. We assume this in the rest of this section. See Section 7.1 for an example wherein reciprocity fails over $Q(q)$.

Fix any right cell, i.e., an irreducible right $\mathcal{B}$-subrepresentation within U^α. Let us denote it by $T_{q,\alpha,R}$ again. Fix a maximal composition series as a $\mathcal{B}(1)$-module of the specialization $T_{q,\alpha,R}(1)$ of $T_{q,\alpha,R}$ at $q = 1$:

$$\hat{T}_{\alpha,0} \subset \hat{T}_{\alpha,1} \subset \cdots \subset \hat{T}_{\alpha,l(\alpha)} = T_{q,\alpha,R}(1).$$

Let $\{x_i\}$ denote the upper canonical basis of X_q as an H_q-module.

Conjecture 6.1 (Nonstandard refined reciprocity)

There exists a basis Z_α of $T_{q,\alpha,R}$ for each α with a filtration

$$Z_{\alpha,0} \subset Z_{\alpha,1} \subset \cdots \subset Z_{\alpha,l(\alpha)} = Z_\alpha,$$

such that:

1. *The specialization $Z_{\alpha,i}(1)$ of $Z_{\alpha,i}$ at $q = 1$ is a basis of $\hat{T}_{\alpha,i}$.*

2. *Let $z^j_{\alpha,i}$ denote the basis elements in $Z_{\alpha,i} \setminus Z_{\alpha,i-1}$. Let $\lambda_{\alpha,i}$ be the partition such that $\hat{T}_{\alpha,i}/\hat{T}_{\alpha,i-1} \cong S_{\lambda_{\alpha,i}}$ as a $\mathcal{B}(1)$-module (or equivalently as an S_r-module). For any α, i, define the left G^H_q-module $W_{q,\alpha,i} = \cup_j X_q^{\otimes r} \cdot z^j_{\alpha,i}$. By the duality conjecture (Conjecture 1.3),*

$$W_{q,\alpha,i} \subseteq W_{q,\alpha} \otimes T_{q,\alpha} \subseteq X_q^{\otimes r}. \tag{14}$$

   We define its specialization $W_{1,\alpha,i}$ at $q = 1$, also denoted by $W_{q,\alpha,i}(1)$, as follows. Let $a(\alpha, i)$ be the largest nonnegative integer such that the limit vector

$$lim_{q\to 1} x_{i_1} \otimes \cdots \otimes x_{i_r}.z^j_{\alpha,i}/(q-1)^{a(\alpha,i)},$$

   is well defined for any $i_1, \ldots, i_r$ and j. We define $W_{1,\alpha,i}$ to be the span of such limits at $q = 1$. Then, $W_{1,\alpha,i}$ is a left G-module contained within the component $V_{\lambda_{\alpha,i}} \otimes S_{\lambda_{\alpha,i}} \subseteq X^{\otimes r}$ in (12).

3. *For any fixed partition λ,*

$$\bigoplus_\alpha \bigoplus_i W_{1,\alpha,i} = V_\lambda \otimes S_\lambda \subseteq X^{\otimes r}, \tag{15}$$

   where, for a given α, i ranges over all indices such that $\lambda_{\alpha,i} = \lambda$.

Furthermore, it may be conjectured that the canonical basis of $T_{q,\alpha,R}$ in terms of the $\mathcal{P}$-monomials as defined in [GCT8] has this property–this would make everything in the conjecture above explicit.

The refined reciprocity conjecture basically says that there is no information loss in the nonstandard setting despite the lack of flatness. In fact, it can be thought of as a variant of flatness.

7 Evidence for duality and reciprocity

Here we describe some concrete computations carried out in MATLAB/Maple that support duality and reciprocity conjectures.

Notation: We denote the q-Weyl module of G_q for a partition λ by $V_{q,\lambda}(G_q)$. We denote $V_{q,\lambda}(GL_q(\mathbb{C}^n))$ by $V_{q,\lambda}(n)$.

7.1 Example 1

Let $H = GL(\mathbb{C}^2)$, $\mathcal{H} = gl(\mathbb{C}^2)$, $X = V_{(3)}(H)$ is its four dimensional irreducible representation, and $G = GL(X) = GL(\mathbb{C}^4)$. Then $H_q = GL_q(\mathbb{C}^2)$, $G_q = GL_q(\mathbb{C}^4)$, and $\mathcal{H}_q = gl_q(\mathbb{C}^2)$. We shall verify duality and reciprocity in this case for $r = 3$. This example is interesting because, as shown in [BZ], the degree three component $\wedge_q^{H,3}[X]$ of the braided exterior algebra vanishes in this case. We expect that the results in this section can be extended to any irreducible representation X of H. But we shall confine ourselves to the case $\dim(X) = 4$, since this seems to be the gist.

Let $\hat{R} = \hat{R}^H_{X,X}$ be the $\hat{R}$-matrix associated with X_q. Let $P = P^H_{X,X}$ and $Q = Q^H_{X,X}$ be the projections on the eigenspaces in $X_q \otimes X_q$ for the positive and negative eigenvalues of $\hat{R}^H_{X,X}$, respectively. Let $x_i = f^i x_0$, where f is the usual operator in $\mathcal{H}_q$, and x_0 is the highest weight vector in X_q. Matrices of $\hat{R}, P$ and Q in the basis $x_i \otimes x_j$ of $X_q \otimes X_q$ can be calculated from the known explicit formulae; cf. [Kass, Kli]. The eigenvalues of $\hat{R}$ turn out to be $q^{9/2}, -q^{-3/2}, q^{-11/2}$ and $-q^{-15/2}$. Explicit matrix of P in the basis $x_i \otimes x_j$ of $X_q \otimes X_q$ is given by

$$P = \frac{1}{f}\mathcal{P}, \tag{16}$$

where

$$f = (q^4 + 1)(q^4 - q^2 + 1)(q^2 + 1)/q^5 \tag{17}$$

and the matrix of $\mathcal{P}$ is as specified in Figure 1 with the following sparse representation: the entry (j, v) in the i-row in Figure 1 means $\mathcal{P}(i, j) = v$. Thus the entry $(5, (q^4 + 1)/q^2)$ in the second row there means $\mathcal{P}(2, 5) = (q^4+1)/q^2$. The entries of $\mathcal{P}$-matrix not shown in Figure 1 are all zero. The scaling factor f here is chosen so that the entries of $\mathcal{P}$-matrix are polynomials in q and q^{-1}. Explicit matrix of

$$\mathcal{Q} = fQ \tag{18}$$

is similar.

7.1.1 Explicit presentation of $\mathcal{B}$

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the $\mathcal{P}$ operators on the first two and the last two factors $X^{\otimes 3}$, respectively; $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined similarly. We have the trivial relations:

$$\mathcal{Q}_i^2 = f\mathcal{Q}_i, \quad \text{and } \mathcal{P}_i^2 = f\mathcal{P}_i.$$

The first nontrivial basic relation among $\mathcal{Q}_i$'s, as determined with the help of a computer, is:

$$\sum_\sigma a_\sigma \mathcal{Q}_\sigma = 0, \tag{19}$$

where σ ranges over the various strings of 1's and 2's as shown in Figure 2, $a_\sigma \in Q[q, q^{-1}]$ are as specified there, and, for a string $\sigma = i_1 i_2 \cdots$, $\mathcal{Q}_\sigma$ denotes the monomial $\mathcal{Q}_{i_1}\mathcal{Q}_{i_2}\cdots$. The second relation is obtained from this by simply interchaning $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Simialrly, the first nontrivial basic relation among $\mathcal{P}_i$'s is

$$\sum_\sigma b_\sigma \mathcal{P}_\sigma = 0, \tag{20}$$

where σ ranges over strings of 1's and 2's as in Figures 3-4, b_σ's are as shown there, and $\mathcal{P}_\sigma$ is defined similarly. The second relation is obtained from this by simply interchanging $\mathcal{P}_1$ and $\mathcal{P}_2$. All coefficients in Figures 2-4 as well as other figures in this section are shown in factored forms, i.e., as products of irreducible polynomials. One may ask if these coefficients have a nice interpretation; we shall turn to this question in Section 7.1.7.

Let $\mathcal{B} = \mathcal{B}_3^H(q)$ be the nonstandard algebra in this case, as defined in Section 5. It is isomorphic to the algebra generated by $\mathcal{P}_i$'s subject to the two basic nontrivial relations among $\mathcal{P}_i$'s described above and the trivial relations $\mathcal{P}_i^2 = f\mathcal{P}_i$, or equivalently, to the algebra generated by $\mathcal{Q}_i$'s subject to the two basic nontrivial relations among $\mathcal{Q}_i$'s described above, and the trivial relations $\mathcal{Q}_i^2 = f\,\mathcal{Q}_i$.

It is clear from these basic defining relations that $\{\mathcal{P}_\sigma\}$ or $\{\mathcal{Q}_\sigma\}$, where σ ranges over all strings of 1's and 2's of length at most 10 without consecutive 1's or 2's, is a basis of $\mathcal{B}$. Its dimension is 21.

7.1.2 Wederburn structure decomposition

Unlike for the Hecke algebras, for the complete Wederburn structure decomposition as in (13) to hold for $\mathcal{B}$, the base field has to contain the algebraic extension K of $Q(q)$ defined as follows. Let

$$disc = \left(5\,q^{16} + 8\,q^{12} - 4\,q^{10} + 18\,q^8 - 4\,q^6 + 8\,q^4 + 5\right)\left(q^8+1\right)^2 q^{24}, \quad (21)$$

and

$$x = disc^{1/2}.$$

Since $disc$ is not a square, x does not belong to $Q(q)$. Let $K = Q(q)[x]$ be the algebraic extension of $Q(q)$ obtained by adjoining x. We assume that $\mathcal{B}$ is defined over this base field. It was found by computer that $\mathcal{B}$ has one one-dimensional irreducible representation T_0, and five two-dimensional irreducible representations T_i, $1 \le i \le 5$, with a complete Wederburn structure decomposition

$$\mathcal{B} = \bigoplus_i U_i, \quad U_i = T_{i,L} \otimes T_{i,R}, \qquad (22)$$

where the basis elements of the various $\mathcal{B}\otimes\mathcal{B}$-bimodules U_i and the explicit representation matrices of the irreducible $\mathcal{B}$-representations T_i are as follows.

Let U_0 be the K-span of $u_0 \in \mathcal{B}$, where u_0 is as specified in Figures 5-6. The coefficients in these and the following figures are in the basis $\{\mathcal{Q}_\sigma\}$. Let U_i, $1 \le i \le 5$, be the K-span of the entries $u_i^1, u_i^{12}, u_i^{21}, u_i^2 \in \mathcal{B}$ of the matrix

$$u_i = \begin{bmatrix} u_i^1 & u_i^{12} \\ u_i^{21} & u_i^2 \end{bmatrix},$$

where u_1^1 is as specified in Figure 7, u_2^1 the element obtained from u_1^1 by substituting $-x$ for x, and u_3^1, u_4^1, u_5^1 as specified in Figures 8-10. Let u_i^2, $1 \le i \le 5$, be the element obtained from u_i^1 by interchanging $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Let $u_i^{12} = u_i^1 \mathcal{Q}_2$, and $u_i^{21} = \mathcal{Q}_2 u_i^1$, for $1 \le i \le 5$.

Then it can be shown that each U_i has a left and right action of $\mathcal{B}$, and as a $\mathcal{B}\otimes\mathcal{B}$-bimodule

$$\mathcal{B} = \bigoplus_i U_i. \qquad (23)$$

The columns of u_i correspond to the left cells and the rows to right cells; i.e., the span of each column (row) is a left (resp. right) $\mathcal{B}$-module, which we shall denote by $T_{i,L}$ (resp. $T_{i,R}$). Thus,

$$\mathcal{B} = \bigoplus_i T_{i,L} \otimes T_{i,R}. \tag{24}$$

Here T_0, the span of u_0, is the trivial one dimensional representation of $\mathcal{B}$, since it can be verified that:

$$\mathcal{Q}_j u_0 = 0, \quad \text{for } j = 1, 2.$$

The representation matrices M_i^1 and M_i^2 of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the basis $\{u_i^1, u_i^{21}\}$ of $T_{i,L}$, $1 \le i \le 5$, are as follows:

$$M_i^1 = \begin{bmatrix} 0 & 1 \\ 0 & f \end{bmatrix},$$

where f is the scaling factor in (16),

$$M_i^2 = \begin{bmatrix} f & g_i \\ 0 & 0 \end{bmatrix},$$

where g_i are as shown in Figure 11; g_2 is obtained from g_1 by substituting $-x$ for x.

Let $T_i(1)$ denote the specialization of T_i at $q = 1$. It is a representation of $\mathcal{B}(1)$, the specialization of $\mathcal{B}$ at $q = 1$. Then $T_0(1)$ corresponds to the trivial one-dimensional representation of S_3. There is no one dimensional representation of $\mathcal{B}$ that specializes to the alternating (signed) one dimensional representation of S_3. This implies that the degree three component $\wedge_q^{H,3}[X]$ of the braided exterior algebra $\wedge_q^H[X]$ in this case is zero–as was already observed and proved by other means in [BZ].

At $q = 1$, the values of $f = f(q)$ and $g_i = g_i(q)$ are as follows:

$$f(1) = g_1(1) = g_3(1) = g_4(1) = g_5(1) = 4, \text{ and } \ g_2(1) = 16.$$

Hence the $\mathcal{B}(1)$-modules $T_1(1), T_3(1), T_4(2)$ and $T_5(2)$ are all isomorphic, and it can be verified that they are isomorphic to the Specht module $S_{(2,1)}$ of the symmetric group S_3 for the partition $(2,1)$. The module $T_2(1)$ is reducible. Because it can be verified that it contains an irreducible $\mathcal{B}(1)$-module $T_2^1(1)$ isomorphic to the trivial one dimensional Specht module $S_{(3)}$ of the symmetric group S_3, and the quotient $T_2^2(1) = T_2(1)/T_2^1(1)$ is isomorphic to the one dimensional signed representation $S_{(1,1,1)}$ of S_3. But $T_2(1)$ is not completely reducible as a $\mathcal{B}(1)$ module. That is, $T_2(1) \not\cong T_2^1(1) \oplus T_2^2(1)$, since it does not contain a submodule isomorphic to $S_{(1,1,1)}$. Thus, though $\mathcal{B}$ is semisimple for generic q, its specialization $\mathcal{B}(1)$ is not semisimple.

7.1.3 Duality

Pick an element u_i from each U_i, $1 \leq i \leq 5$; say, $u_i = u_i^1$, and u_0 is as before. For $0 \leq i \leq 5$, let $W_i = X_q^{\otimes 3} \cdot u_i$, which has a left action of the nonstandard quantum group G_q^H. These are nonisomorphic irreducible representations of G_q^H. Their explicit decompositions as H_q-modules, $H_q = GL_q(\mathbb{C}^2)$, were determined with the help of computer. They are as follows.

The module W_0 is isomorphic to the sixteen dimensional degree three component $\mathbb{C}_q^{H,3}[X]$ of the braided symmetric algebra [BZ] with the following decomposition as an H_q-module:

$$W_0 = V_{q,(9)}(2) \oplus V_{q,(7,2)}(2);$$

recall that $V_{q,\lambda}(n)$ denotes the q-Weyl module of $GL_q(n)$ corresponding to the partition λ. This decomposition of $\mathbb{C}_q^{H,3}[X]$ in this case agrees with the one obtained in [BZ] by other means.

The modules W_i, $i > 0$, are distinct irreducible representations of G_q^H with the following decompositions as H_q-modules:

$$\begin{array}{rcl} W_1 & \cong & V_{q,(6,3)}(2), \\ W_2 & \cong & V_{q,(6,3)}(2), \\ W_3 & \cong & V_{q,(8,1)}(2), \\ W_4 & \cong & V_{q,(5,4)}(2), \\ W_5 & \cong & V_{q,(7,2)}(2). \end{array} \tag{25}$$

Their dimensions are $4, 4, 8, 2$ and 6, respectively. Though W_1 and W_2 are isomorphic as H_q-modules, they are nonisomorphic as G_q^H-modules; the matrix coefficients of W_2 are obtained from those for W_1 by substituting $-x$ for x.

It can be verified that, as a $G_q^H \times \mathcal{B}$-bimodule,

$$X_q^{\otimes 3} \cong \oplus_i W_i \otimes T_i, \tag{26}$$

as per the duality conjecture (Conjecture 1.3).

7.1.4 Reciprocity

Let m^i_μ denote the multiplicity of the Specht module S_μ of the symmetric group S_3 in the $\mathcal{B}(1)$-module T_i. Then, we see that

$$\begin{aligned}
&m^0_{(3)} = 1,\\
&m^1_{(2,1)} = m^3_{(2,1)} = m^4_{(2,1)} = m^5_{(2,1)} = 1,\\
&m^2_{(3)} = m^2_{(1,1,1)} = 1.
\end{aligned}$$

Furthermore, it can be verified that the various G_q-modules, $G_q = GL_q(\mathbb{C}^4)$, decompose as follows when considered as H_q-modules:

$$\begin{aligned}
V_{q,(3)}(4) &\cong m^0_{(3)} W_0 \oplus m^2_{(3)} W_2,\\
&\cong V_{q,(9)}(2) \oplus V_{q,(7,2)}(2) \oplus V_{q,(6,3)}(2), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
V_{q,(2,1)}(4) &\cong m^1_{(2,1)} W_1 \oplus m^3_{(2,1)} W_3 \oplus m^4_{(2,1)} W_4 \oplus m^5_{(2,1)} W_5\\
&\cong V_{q,(6,3)}(2) \oplus V_{q,(8,1)}(2) \oplus V_{q,(5,4)}(2) \oplus V_{q,(7,2)}(2).
\end{aligned}$$

This verifies the nonstandard reciprocity conjecture (Conjecture 1.4) in this case.

7.1.5 Refined reciprocity

Fix a right cell within U_2 isomorphic to the representation $T_{2,R}$; say, the one spanned by u^1_2 and u^{12}_2. We shall denote it by $T_{2,R}$ again. Let $z_0 \in T_{2,R}$ be the element such that $z_0 \mathcal{Q}_2 = 0$. Its coefficients are shown in Figure 12 in the basis $\{Q_\sigma\}$. Let $z_1 = u^1_2$. Then the basis $Z = \{z_0, z_1\}$ of $T_{2,R}$ admits a filtration

$$Z_0 = \{z_0\} \subseteq Z_1 = \{z_0, z_1\},$$

that yields at $q = 1$ a composition series of $T_{2,R}(1)$ as a $\mathcal{B}(1)$-module:

$$\hat{T}_{2,0} \subset \hat{T}_{2,1} = T_{2,R}(1),$$

where $\hat{T}_{2,0}$, spanned by the specialization $z_0(1)$ of z_0, is the one dimensional trivial representation of S_3, and $\hat{T}_{2,1}/\hat{T}_{2,0}$ is the one-dimensional signed representation of S_3.

Let $W_{2,1} = X_q^{\otimes 3} \cdot z_1$ and $W_{2,0} = X_q^{\otimes 3} \cdot z_0$ be the G_q^H-submodules of $X_q^{\otimes 3}$, and $W_{2,1}(1), W_{2,0}(1)$ their specializations at $q = 1$. It can be verified that at $q = 1$ we get:

$$W_{2,1}(1) = \wedge^3(X) \subseteq X^{\otimes 3}, \quad \text{and} \quad W_0(1) \oplus W_{2,0}(1) = \mathrm{Sym}^3(X) \subseteq X^{\otimes 3}, \tag{27}$$

where $\wedge^3(X)$ and $\mathrm{Sym}^3(X)$ are the Weyl modules of $G = GL(X)$ for the partitions $(1,1,1)$ and (3), respectively, and $W_0(1)$ the specialization of W_0 at $q = 1$.

For example, Figures 13-16 show the nonzero coefficients of the elements $a = (x_1 \otimes x_2 \otimes x_0) \cdot z_1$ and $b = (x_1 \otimes x_2 \otimes x_0) \cdot z_0$ in the monomial basis $\{x_i \otimes x_j \otimes x_k\}$ of $X_q^{\otimes 3}$. It can be verified that the specialization $a(1)$ at $q = 1$ of a indeed belongs to the subspace $\wedge^3(X) \subseteq X^{\otimes 3}$. The specialization $b(1)$ of b, as it is, just vanishes, since its coefficients are divisible by $(q-1)^2$. But instead we consider the basis element $b' = b/(q-1)^2$ of $W_{2,0}$. Then its specialization $b'(1)$ at $q = 1$ indeed belongs to the subspace $\mathrm{Sym}^3(X)$ of $X^{\otimes 3}$. The equation (27) can be verified similarly.

Similarly it can be verified that

$$\lim_{q \to 1} \bigoplus_{i=1,3,4,5} (X_q^{\otimes 3} \cdot u_i^1 \cup X_q^{\otimes 3} \cdot u_i^{12}) = V_{(2,1)} \otimes S_{(2,1)} \subseteq X^{\otimes 3}.$$

This verifies the refined reciprocity conjecture in this case. In particular, it explains what happens to the exterior and symmetric algebra components here. Specifically, though the braided exterior algebra component $\wedge_q^{H,3}[X] = 0$,

$$W_{2,1}(1) = \wedge^{H,3}[X].$$

Thus the q-deformation of $\wedge_3^H[X]$ has simply relocated itself as $W_{2,1}$ in the decomposition

$$X_q^{\otimes 3} = \oplus W_i \otimes T_i.$$

Similarly, the symmetric algebra component $\mathbb{C}^{H,3}[X]$ splits in two parts, and the q-deformations of these parts, namely W_0 and $W_{2,0}$, get distributed in this decomposition. The situation for $V_{2,1}$ is similar. Thus, overall, there is no information loss; the information has only been redistributed. As per the refined reciprocity conjecture, this is a general phenomenon.

7.1.6 Base field $Q(q)$

Let us now see what happens if the base field is $Q(q)$ instead. The $\mathcal{B}$-representations T_0, T_3, T_4, T_5 are already defined over $Q(q)$. But T_1 and T_2 merge into a four dimensional $\mathcal{B}$-representation T_{12} defined over $Q(q)$.

Explicitly, it can be realized within $\mathcal{B}$ as the linear span of the elements

$$\begin{aligned} v1 &= (u_1^1 + u_2^1)/2, \\ v2 &= (u_1^1 - u_2^1)/(2x), \\ v3 &= (u_1^{21} + u_2^{21})/2, \\ v3 &= (u_1^{21} - u_2^{21})/(2x). \end{aligned}$$

Representation matrices of left multiplication by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in the basis $\{v_i\}$ are, respectively,

$$M^1 = \begin{bmatrix} \frac{q^{10}+q^6+q^4+1}{q^5} & 0 & a & 1/2\,q^{-20} \\ 0 & \frac{q^{10}+q^6+q^4+1}{q^5} & b & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with

$$\begin{aligned} a &= 1/2\,\frac{3\,q^{16}+4\,q^{12}-2\,q^{10}+10\,q^8-2\,q^6+4\,q^4+3}{q^8}, \\ b &= 1/2\,q^4(5\,q^{32}+8\,q^{28}-4\,q^{26}+28\,q^{24}-4\,q^{22}+24\,q^{20}-8\,q^{18}+46\,q^{16} \\ &\quad -8\,q^{14}+24\,q^{12}-4\,q^{10}+28\,q^8-4\,q^6+8\,q^4+5), \end{aligned}$$

and

$$M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & \frac{q^{10}+q^6+q^4+1}{q^5} & 0 \\ 0 & 1 & 0 & \frac{q^{10}+q^6+q^4+1}{q^5} \end{bmatrix}.$$

Similarly, the G_q^H-modules W_0, W_3, W_4, W_5 are already defined over $Q(q)$. The modules W_1 and W_2 merge into an eight-dimensional G_q^H-module $W_{12} \cong X_q^{\otimes 3} \cdot v_i$, for any i–this is defined over $Q(q)$. As an H_q-module,

$$W_{1,2} \cong 2 \cdot V_{q,(6,3)}(2).$$

A variant of the duality also holds. Specifically, the components $W_1 \otimes T_1$ and $W_2 \otimes T_2$ in the decomposition (26) of $X_q^{\otimes 3}$ merge into one sixteen dimensional $G_q^H \times \mathcal{B}$-bimodule defined over $Q(q)$. As a G_q^H module, it is a direct sum of two copies of W_{12}, and as a $\mathcal{B}$-module a direct sum of four copies of T_{12}. But for the reciprocity to hold, the base field has to be $K = Q(q)[x]$ as before or larger. Indeed, it can be seen here that the reciprocity conjecture fails over the base field $Q(q)$. This illustrates the need for base extension in general.

It may be illuminating to compare the $r = 3$ case here with the one for the Kronecker problem treated in [GCT4]. The one here is basically a more complex version of the one in [GCT4], because the basic defining relations here (Figures 2-4) are more complex versions of the ones in [GCT4].

7.1.7 On $r > 3$ and positivity

Similar symbolic computations for $r = 4$ seem beyond the reach of desktop MATLAB/Maple. Fortunately, this case for the Kronecker problem is within the reach, and will be treated in the next section. The $r = 4$ case, $H = GL_2(\mathbb{C})$, X four dimensional, is expected to be its more complex version just as for $r = 3$.

But it does not seem possible to progress much beyond $r = 3$ using the brute force computer-based approach that we are following here. What is neeeded is an explicit presentation for $\mathcal{B}_r^H$ akin to the explicit presentation for the Hecke algebra, or the one for $r = 3$ in Section 7.1.1. That is, we need an explicit set of generating relations among $\mathcal{Q}_i$ or $\mathcal{P}_i$'s, each of the form

$$\sum a_\sigma \mathcal{Q}_\sigma = 0, \tag{28}$$

or

$$\sum b_\sigma \mathcal{P}_\sigma = 0, \tag{29}$$

where $\mathcal{Q}_\sigma$ and $\mathcal{P}_\sigma$, for a string $\sigma = i_1 i_2 \cdots$ of symbols in $\{1, \cdots, r-1\}$, denote the monomials $\mathcal{Q}_{i_1}\mathcal{Q}_{i_2}\cdots$ and $\mathcal{P}_{i_1}\mathcal{P}_{i_2}\cdots$, respectively, and each a_σ and b_σ has an explicit interpretation (formula).

The coefficients a_σ and b_σ in Figures 2-4 for the $r = 3$ case do not seem to have any obvious elementary interpretation. Hence, in general, one can only expect nonelementary interpretations for the coefficients a_σ and b_σ in (28)-(29). The following numerical analysis of these coefficients for the $r = 3$ case suggests that $\mathcal{B}_r^H$, in general, may plausibly have an explicit presentation, the coefficients a_σ and b_σ of whose generating relations have nonelementary interpretations in the spirit of the one for Kazhdan-Lusztig polynomials. By this we mean that each a_σ has an explicit formula of the form of an alternating sum

$$a_\sigma = (-1)^{d(\sigma)}(q^{1/2} - q^{-1/2})^{d'(\sigma)}(\sum_{j=0}^{s(\sigma)}(-1)^j a_\sigma^j), \tag{30}$$

for some nonnegative integers $d(\sigma), d'(\sigma), s(\sigma)$, where

1. $s(\sigma)$ is small, say bounded by a polynomial of a fixed degree in r and $\dim(X)$ in the present case when $H = GL_2(\mathbb{C})$, and in r, the rank of H and the size of μ in the general plethysm problem (Problem 1.1),

2. each a_σ^j is a $-$-invariant (note that a_σ is $-$-invariant), positive and unimodal polynomial in q and q^{-1}; positive means each coefficient of a_σ^j is nonnegative, and unimodal means, if $a_\sigma^j(-k), \ldots, a_\sigma^j(k)$ are the coefficients of a_σ^j, then

$$a_\sigma^j(-k) \leq a_\sigma^j(-k+1) \leq \cdots \leq a_\sigma^j(-1) \leq a_\sigma^j(0) \leq a_\sigma^j(1) \leq \cdots a_\sigma^j(k),$$

3. each $a_\sigma^j(s)$ has a topological interpretation akin to that for Kazhdan-Lusztig polynomials, i.e., as the rank of an appropriate cohomology group. Then the duality $a_\sigma^j(-s) = a_\sigma^j(s)$ as per the $-$-invariance of a_σ^j should come out as a consequence of some form of Poincare duality and the unimodality as a consequence of some form of Hard Lefschetz,

and each b_σ has a similar explicit formula of the form

$$b_\sigma = (-1)^{\bar{d}(\sigma)}(q^{1/2} - q^{-1/2})^{\bar{d}'(\sigma)}(\sum_{j=0}^{\bar{s}(\sigma)}(-1)^j b_\sigma^j). \tag{31}$$

We shall call such an interpretation for a_σ or b_σ, if it exists, a *positive, unimodal, and topological* interpretation.

Ideally speaking, one would like each $s(\sigma)$ and $\bar{s}(\sigma)$ above to be zero, but this may not always be possible for the reasons given below. It is plausible that there exists some notion of cohomological depth that measures the extent of nonflatness, and which provides an upper bound on $s(\sigma)$ and $\bar{s}(\sigma)$ in such a topological interepretation, if it exists. For example, in the Kronecker problem, the braided symmetric and exterior algebras $\mathbb{C}_q^H[X]$ and $\wedge_q^H[X]$ are flat deformations of the classical algebras $\mathbb{C}[X]$ and $\wedge[X]$. In this case, one can expect an explicit presentation for $\mathcal{B}_q^H$ whose coefficients a_σ and b_σ have positive topological interpretation with $s(\sigma), \bar{s}(\sigma) = 0$ in (30) and (31). This is because a_σ and b_σ here are akin to the structural constants $c_{b,b'}^{b''}$, $d_{b''}^{b,b'}$ in Theorem 4.4, which occur in the defining Laplace relations for G_q^H, and which, in the Kroncker problem, are conjecturally polynomials in q and q^{-1} with nonnegative coefficients for the reasons indicated there. But in general when $\mathbb{C}_q^H[X]$ and $\wedge_q^H[X]$ are nonflat deformations, such cohomological depth would not vanish, and hence $s(\sigma)$ and $\bar{s}(\sigma)$ may be nonzero, but still small as indicated above.

We now turn to the analysis of the coefficients in the $r = 3$ case mentioned above which suggests that such an interpretation may plausibly exist. First let us oberve that the scaling factor f in (17) used in the analysis so far is formally not the correct scaling factor. To get the latter, we have to look at the formal expressions for P and Q in terms of $\hat{R}$. Since the eigenvalues of $\hat{R}$ in the present case are

$$q_1 = q^{9/2}, \quad q_2 = -q^{-3/2}, \quad q_3 = q^{-11/2}, \quad \text{and} \quad q_4 = -q^{-15/2},$$

we have

$$P = \frac{(\hat{R} - q_2)(\hat{R} - q_3)(\hat{R} - q_4)}{(q_1 - q_2)(q_1 - q_3)(q_1 - q_4)} + \frac{(\hat{R} - q_1)(\hat{R} - q_2)(\hat{R} - q_4)}{(q_3 - q_1)(q_3 - q_2)(q_3 - q_4)}, \tag{32}$$

and

$$Q = \frac{(\hat{R} - q_1)(\hat{R} - q_3)(\hat{R} - q_4)}{(q_2 - q_1)(q_2 - q_3)(q_2 - q_4)} + \frac{(\hat{R} - q_1)(\hat{R} - q_2)(\hat{R} - q_3)}{(q_4 - q1)(q_4 - q_2)(q_4 - q_3)}. \tag{33}$$

Hence, formally we should have defined the rescaled versions $\mathcal{P}$ and $\mathcal{Q}$ of P and Q by the equations

$$\mathcal{P} = f_p P, \quad f_p = (q_1 - q_2)(q_1 - q_3)(q_1 - q_4)(q_3 - q_1)(q_3 - q_2)(q_3 - q_4), \tag{34}$$

and

$$\mathcal{Q} = f_q Q, \quad f_q = (q_2 - q_1)(q_2 - q_3)(q_2 - q_4)(q_4 - q_1)(q_4 - q_3)(q_4 - q_2), \tag{35}$$

instead of the equations (16) and (18). The scaling factor f in (17) was the smallest factor chosen so that the matrix coefficients of P and Q after rescaling become polynomials in q, q^{-1}. But this choice was dependent on the accidental cancellations in the numerators and denominators in (32) and (33). The choice of scaling makes no essential difference in Sections 7.1.1-7.1.6. But it does matter in the study of positivity below.

Hence, let us redefine $\mathcal{P}$ and $\mathcal{Q}$ as per (34) and (35). Let us denote the coefficients of the old defining relations (19) and (20) among $\mathcal{Q}_i$'s and $\mathcal{P}_i$'s by a'_σ and b'_σ, and the coefficients of the defining relations among the new $\mathcal{Q}_i$'s and $\mathcal{P}_i$'s by a''_σ and b''_σ. Then we have

$$a''_\sigma = (\frac{-(q-1)^2}{q})^{11-l(\sigma)}\bar{a}_\sigma, \quad with \quad \bar{a}_\sigma = (\hat{f}_q)^{11-l(\sigma)}a'_\sigma,$$

and

$$b''_\sigma = (\frac{-(q-1)^2}{q})^{11-l(\sigma)}\bar{b}_\sigma, \quad \text{with} \qquad \bar{b}_\sigma = (\hat{f}_p)^{11-l(\sigma)}b'_\sigma,$$

where $l(\sigma)$ denotes the length of σ,

$$\begin{aligned} \hat{f}_p &= \tfrac{-q}{(q-1)^2}\tfrac{f_p}{f} \\ &= 10+8\,q+2\,q^4+12\,q^{-1}+18\,q^{-6}+6\,q^2+4\,q^3+q^5 \\ &\quad +14\,q^{-2}+18\,q^{-4}+16\,q^{-12}+16\,q^{-8}+q^{-27}+16\,q^{-11} \\ &\quad +16\,q^{-10}+17\,q^{-7}+6\,q^{-24}+18\,q^{-5}+2\,q^{-26}+16\,q^{-3} \\ &\quad +16\,q^{-9}+10\,q^{-22}+14\,q^{-20}+4\,q^{-25}+8\,q^{-23}+12\,q^{-21}+18\,q^{-18} \\ &\quad +16\,q^{-19}+18\,q^{-16}+18\,q^{-17}+17\,q^{-15}+16\,q^{-14}+16\,q^{-13}, \end{aligned}$$

and

$$\begin{aligned} \hat{f}_q &= \tfrac{-q}{(q-1)^2}\tfrac{f_q}{f} \\ &= 10\,q^{-12}+2\,q^{-8}+q^{-31}+8\,q^{-27}+8\,q^{-11}+6\,q^{-10}+q^{-7} \\ &\quad +2\,q^{-30}+10\,q^{-24}+6\,q^{-28}+10\,q^{-26}+4\,q^{-9}+6\,q^{-22} \\ &\quad +2\,q^{-20}+10\,q^{-25}+8\,q^{-23}+4\,q^{-29}+4\,q^{-21}+2\,q^{-18} \\ &\quad +2\,q^{-19}+6\,q^{-16}+4\,q^{-17}+8\,q^{-15}+10\,q^{-14}+10\,q^{-13}. \end{aligned}$$

Both f_p and f_q are positive polynomials.

Let us define

$$\begin{aligned} \hat{a}_\sigma &= \hat{f}_q^2 a'_\sigma, \quad \text{for} \quad \sigma = 121212121, \\ &= \hat{f}_q a'_\sigma, \quad \text{otherwise}, \end{aligned} \tag{36}$$

and

$$\begin{aligned} \hat{b}_\sigma &= b'_\sigma, \qquad \text{for} \quad \sigma = \emptyset, \text{and } 2, \\ &= \hat{f}_p b'_\sigma, \quad \text{otherwise}. \end{aligned} \tag{37}$$

Since f_p and f_q are positive, the positivity properties of $\hat{a}_\sigma$ and $\bar{a}_\sigma$ (also $\hat{b}_\sigma$ and $\bar{b}_\sigma$) are similar; it turns out that the unimodularity properties are also similar. Hence we shall focus on $\hat{a}_\sigma$ and $\hat{b}_\sigma$ in what follows. Since $\hat{a}_\sigma$ is $-$-invariant, it is of the form $\hat{a}_\sigma(0) + \sum_{t>0} \hat{a}_\sigma(t)(q^t + q^{-t})$. Let $\hat{A}_\sigma$ be the vector $[a_\sigma(0), a_\sigma(1), \ldots]$; the vector $\hat{B}_\sigma$ is defined similarly. Figure 17 shows $\hat{A}_\sigma$ for the various σ in Figure 2; the vector for each σ is obtained by concatenating the rows in front of that σ. Figures 18-20 similarly show $\hat{B}_\sigma$ for the various σ in Figures 3-4; only the distinct $\hat{B}_\sigma$'s are shown. It may be seen the $\hat{A}_\sigma$'s are positive and nonincreasing. Thus all a_σ are positive and unimodal, and hence, of the form (30) with $s(\sigma) = 0$. All $\hat{B}_\sigma$'s are positive and nonincreasing, except for $\sigma = 121, 1212$ and 21212, for which each $\hat{B}_\sigma$ is positive and unimodal except at the tail. Thus all b_σ, for $\sigma \neq 121, 1212, 21212$, are positive and unimodal, and hence of the form (31) with $\bar{s}(\sigma) = 0$. For $\sigma = 121, 1212, 21212$, b_σ seems to be of the form (31) with $\bar{s}(\sigma) = 1$, both b^0_σ and b^1_σ being positive and unimodal, b^0_σ being the dominant polynomial that accounts for b_σ's mostly positive and unimodal behaviour, and b^1_σ the error polynomial that accounts for the deviation at the tail.

The (co)multiplicative structural constants $c^{b''}_{b,b'}$ and $d^{b,b'}_{b''}$ for the canonical basis of the braided exterior algebra $\wedge^{H,r}_q[X]$, which occur in the Laplace relations for the general nonstandard quantum group G^H_q (cf. Theorem 4.5), are akin to the structure constants a_σ and b_σ in (28) and (29). Hence, we can expect a similar positive topological interpretation for $c^{b''}_{b,b'}$ and $d^{b,b'}_{b''}$ (but not necessarily unimodality since $c^{b''}_{b,b'}$ and $d^{b,b'}_{b''}$ need not be $-$-invariant). The experimental evidence in [GCT8] suggests that the structure constants associated with the canonical bases of the matrix coordinate ring of G^H_q and the ring $\mathcal{B}^H_q$ defined there may also have similar positive topological interpretations (additionally unimodal for $\mathcal{B}^H_q$).

7.2 Example 2

Now we verify the duality and reciprocity conjectures for the special case of the Kronecker problem (Section 4.1), when $H = GL(V) \times GL(W)$, $V = W = \mathbb{C}^2$ and $G = GL(X)$, $X = V \otimes W \cong \mathbb{C}^4$, and $r = 4$. Thus $G_q = GL_q(\mathbb{C}^4)$, and $H_q = GL_q(\mathbb{C}^2) \times GL_q(\mathbb{C}^2)$. Let $\mathcal{B} = \mathcal{B}^H_r$ be the nonstandard algebra in this case and $P_i = p^{+,H}_{X,i}, Q_i = p^{-,X}_{X,i}$, $i < r$, the positive and negative projection operators as in Section 5. Let $\mathcal{P}_i$ and $\mathcal{Q}_i$ be the rescaled versions of P_i and Q_i as defined in [GCT4]. Then $\mathcal{B}$ is generated by $\mathcal{P}_i$, or equivalently, $\mathcal{Q}_i$. The explicit generating relations among $\mathcal{P}_i$'s and $\mathcal{Q}_i$'s turn out to be very complicated. For example, Figures 21-23 reproduced from [GCT4] shows a typical generating relation among $\mathcal{Q}_i$'s with 74 terms. There are several

$$
\begin{bmatrix}
1, \frac{\left(q^4+1\right)\left(q^4-q^2+1\right)\left(q^2+1\right)}{q^5} & & & \\
2, q\left(q^4+1\right) & 5, \frac{q^4+1}{q^2} & & \\
3, \frac{\left(q^2+1\right)\left(q^8-q^6+q^4-q^2+1\right)}{q^5} & 6, \frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^4} & 9, \frac{q^2+1}{q} & \\
4, \frac{q^8-q^6+q^4+1}{q^3} & 7, \frac{(q-1)^2(q+1)^2\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^6} & 10, \frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^5} & 13, \frac{q^4+1}{q^2} \\
2, \frac{q^4+1}{q^2} & 5, \frac{q^4+1}{q^5} & & \\
3, \frac{\left(q^2+1\right)^2(q-1)(q+1)}{q^2} & 6, 2\,\frac{q^2+1}{q} & 9, -\frac{\left(q^2+1\right)^2(q-1)(q+1)}{q^2} & \\
4, \frac{(q-1)^2(q+1)^2\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^2} & 7, \frac{2\,q^8+q^6-2\,q^2+1}{q^5} & 10, -\frac{q^8-q^6-2\,q^4-q^2+1}{q^4} & 13, -\frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q} \\
8, \frac{\left(q^2+1\right)\left(q^8-q^6+q^4-q^2+1\right)}{q^5} & 11, \frac{\left(q^2+1\right)^2(q-1)(q+1)}{q^4} & 14, \frac{q^2+1}{q} & \\
3, \frac{q^2+1}{q} & 6, -\frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^4} & 9, \frac{\left(q^2+1\right)\left(q^8-q^6+q^4-q^2+1\right)}{q^5} & \\
4, \frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q} & 7, -\frac{q^8-q^6-2\,q^4-q^2+1}{q^4} & 10, \frac{q^8-2\,q^6+q^2+2}{q^3} & 13, \frac{(q-1)^2(q+1)^2\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^2} \\
8, \frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^2} & 11, 2\,\frac{q^2+1}{q} & 14, -\frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^2} & \\
12, q\left(q^4+1\right) & 15, \frac{q^4+1}{q^2} & & \\
4, \frac{q^4+1}{q^2} & 7, -\frac{(q-1)(q+1)\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^5} & 10, \frac{(q-1)^2(q+1)^2\left(q^2+q+1\right)\left(q^2-q+1\right)}{q^6} & 13, \frac{q^8+q^4-q^2+1}{q^5} \\
8, \frac{q^2+1}{q} & 11, -\frac{\left(q^2+1\right)^2(q-1)(q+1)}{q^4} & 14, \frac{\left(q^2+1\right)\left(q^8-q^6+q^4-q^2+1\right)}{q^5} & \\
12, \frac{q^4+1}{q^2} & 15, \frac{q^4+1}{q^5} & & \\
16, \frac{\left(q^4+1\right)\left(q^4-q^2+1\right)\left(q^2+1\right)}{q^5} & & &
\end{bmatrix}
$$

Figure 1: $\mathcal{P}$-matrix

σ	a_σ
1	$-(q^{12}+q^{10}+2\,q^8+2\,q^4+q^2+1)^2(q^8-q^6+q^4-q^2+1)^2(q^2+1)^4(q^4-q^2+1)^4(q^4+1)^2/q^{36}$
121	$(1+7\,q^4+q^2+5\,q^6+18\,q^8+21\,q^{42}-107\,q^{20}+q^{50}-107\,q^{32}+73\,q^{14}+187\,q^{18}-14\,q^{16}+402\,q^{26}$ $-197\,q^{28}+20\,q^{40}+187\,q^{34}+73\,q^{38}-197\,q^{24}+328\,q^{30}+q^{52}+328\,q^{22}+7\,q^{48}+5\,q^{46}+18\,q^{44}$ $+20\,q^{12}+21\,q^{10}-14\,q^{36})(q^2+1)^2(q^4-q^2+1)^2/q^{32}$
12121	$-(1+8\,q^4+3\,q^2+4\,q^6+33\,q^8+12\,q^{42}+80\,q^{20}+3\,q^{50}+80\,q^{32}+27\,q^{14}+113\,q^{18}+115\,q^{16}+360\,q^{26}$ $-9\,q^{28}+76\,q^{40}+113\,q^{34}+27\,q^{38}-9\,q^{24}+253\,q^{30}+q^{52}+253\,q^{22}+8\,q^{48}+4\,q^{46}+33\,q^{44}+76\,q^{12}$ $+12\,q^{10}+115\,q^{36})/q^{26}$
1212121	$(3\,q^{36}+2\,q^{34}+8\,q^{32}+5\,q^{30}+17\,q^{28}+30\,q^{26}+11\,q^{24}$ $+61\,q^{22}-15\,q^{20}+108\,q^{18}-15\,q^{16}+61\,q^{14}+11\,q^{12}+30\,q^{10}+17\,q^8+5\,q^6+8\,q^4+2\,q^2+3)/q^{18}$
121212121	$-(q^{20}+3\,q^{18}+q^{16}+5\,q^{14}-2\,q^{12}+16\,q^{10}-2\,q^8+5\,q^6+q^4+3\,q^2+1)/q^{10}$
12121212121	1

Figure 2: Coefficients of the basic generating relation among $\mathcal{Q}_i$'s

σ	b_σ
$\emptyset$	$-(q^2+1)^5(q^4+1)^3(q^4-q^2+1)^6(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6-q^3+1)$ $\times(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)$ $\times(q^2+q+1)^2(q^2-q+1)^2(q-1)^4(q+1)^4/q^{51}$
2	$(q^2+1)^4(q^4-q^2+1)^5(q^4+1)^2(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)$ $\times(q^6-q^3+1)(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)(q^6+q^5+q^4+q^3+q^2+q+1)$ $\times(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+q+1)^2(q^2-q+1)^2(q-1)^4(q+1)^4/q^{46}$
1	$(q^2+1)^4(q^4-q^2+1)^4(q^4+1)^2(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-10\,q^{42}-4\,q^{58}+7\,q^{56}-9\,q^{54}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+2\,q^{60}+13\,q^{44}+13\,q^{12}-12\,q^{10}+q^{36})/q^{46}$
12	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-10\,q^{42}-4\,q^{58}+7\,q^{56}-9\,q^{54}+12\,q^{20}$ $-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}$ $+q^{22}+13\,q^{48}-12\,q^{46}+2\,q^{60}+13\,q^{44}+13\,q^{12}-12\,q^{10}+q^{36})/q^{41}$
21	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-10\,q^{42}-4\,q^{58}+7\,q^{56}-9\,q^{54}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+2\,q^{60}+13\,q^{44}+13\,q^{12}-12\,q^{10}+q^{36})/q^{41}$
212	$(q^2+1)^2(q^4-q^2+1)^2(2+7\,q^4-4\,q^2-9\,q^6+11\,q^8-10\,q^{42}-4\,q^{58}+7\,q^{56}-9\,q^{54}+12\,q^{20}$ $-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}$ $+q^{22}+13\,q^{48}-12\,q^{46}+2\,q^{60}+13\,q^{44}+13\,q^{12}-12\,q^{10}+q^{36})/q^{36}$
121	$(q^2+1)^2(q^4+1)^2(q^4-q^2+1)^2(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-9\,q^{42}+27\,q^{20}$ $-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}$ $+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44}-3\,q^{12}-9\,q^{10}+q^{36})/q^{36}$
1212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}$ $-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}$ $+2\,q^{48}-2\,q^{46}-2\,q^{44}-3\,q^{12}-9\,q^{10}+q^{36})/q^{31}$

Figure 3: The first eight terms of the basic generating relation among $\mathcal{P}_i$'s

σ	b_σ
2121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}$ $-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}$ $+2\,q^{48}-2\,q^{46}-2\,q^{44}-3\,q^{12}-9\,q^{10}+q^{36})/q^{31}$
21212	$(2+2\,q^4-4\,q^2-2\,q^6-2\,q^8-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}$ $+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44}-3\,q^{12}$ $-9\,q^{10}+q^{36})/q^{26}$
12121	$(q^2+1)^2(q^4+1)^2(q^4-q^2+1)^2(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{26}$
121212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{21}$
212121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{21}$
2121212	$(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{16}$
1212121	$-(q^2+1)^2(q^4+1)^2(q^4-q^2+1)^2(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{18}$
12121212	$(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
21212121	$(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
212121212	$-(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^8$
121212121	$(q^2+1)^2(q^4+1)^2(q^4-q^2+1)^2/q^{10}$
1212121212	$-(q^2+1)(q^4+1)(q^4-q^2+1)/q^5$
2121212121	$-(q^2+1)(q^4+1)(q^4-q^2+1)/q^5$
21212121212	1

Figure 4: The last fourteen terms of the basic generating relation among $\mathcal{P}_i$'s

σ	$Coefficient$
$\emptyset$	$(q^4+q^3+q^2+q+1)(q^4-q^3+q^2-q+1)(q^6-q^3+1)(q^6+q^3+1)(q^6-q^5+q^4-q^3+q^2-q+1)$ $\times(q^6+q^5+q^4+q^3+q^2+q+1)(2\,q^8-2\,q^6+3\,q^4-2\,q^2+2)(q^2+q+1)^2(q^2-q+1)^2$ $\times(q^4+1)^2(q-1)^4(q+1)^4(q^2+1)^4(q^4-q^2+1)^5/q^{46}$
2	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})/q^{41}$
1	$-(q^2+1)^3(q^4+1)(q^4-q^2+1)^3(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}$ $+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}$ $+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})/q^{41}$
12	$(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}$ $+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}$ $-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})(q^2+1)^2(q^4-q^2+1)^2/q^{36}$
21	$(2+13\,q^{12}-12\,q^{10}+11\,q^8+7\,q^4-4\,q^2-9\,q^6+q^{36}-10\,q^{42}+12\,q^{20}-12\,q^{50}-11\,q^{32}-12\,q^{14}-10\,q^{18}$ $+13\,q^{16}+16\,q^{26}-11\,q^{28}+12\,q^{40}+16\,q^{34}+q^{38}+q^{24}+28\,q^{30}+11\,q^{52}+q^{22}+13\,q^{48}-12\,q^{46}+13\,q^{44}$ $-4\,q^{58}+7\,q^{56}-9\,q^{54}+2\,q^{60})(q^2+1)^2(q^4-q^2+1)^2/q^{36}$
212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}$ $-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}$ $+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{31}$
121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}$ $-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}$ $-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{31}$
1212	$(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}$ $+q^{16}-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{26}$
2121	$(2-3\,q^{12}-9\,q^{10}-2\,q^8+2\,q^4-4\,q^2-2\,q^6+q^{36}-9\,q^{42}+27\,q^{20}-4\,q^{50}+27\,q^{32}-13\,q^{14}-48\,q^{18}+q^{16}$ $-110\,q^{26}+53\,q^{28}-3\,q^{40}-48\,q^{34}-13\,q^{38}+53\,q^{24}-77\,q^{30}+2\,q^{52}-77\,q^{22}+2\,q^{48}-2\,q^{46}-2\,q^{44})/q^{26}$

Figure 5: First nine coefficients of u_0

σ	$Coefficient$
21212	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8-q^6+q^4+1)(q^8+q^4-q^2+1)/q^{21}$
12121	$-(q^2+1)(q^4+1)(q^4-q^2+1)(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8-q^6+q^4+1)(q^8+q^4-q^2+1)/q^{21}$
121212	$(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{16}$
212121	$(3\,q^{16}+2\,q^{14}+14\,q^8+2\,q^2+3)(q^8+q^4-q^2+1)(q^8-q^6+q^4+1)/q^{16}$
2121212	$(q^2+1)(q^4-q^2+1)(q^4+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
1212121	$(q^2+1)(q^4-q^2+1)(q^4+1)(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^{13}$
12121212	$-(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^8$
21212121	$-(3\,q^{16}-q^{14}+3\,q^{12}-3\,q^{10}+12\,q^8-3\,q^6+3\,q^4-q^2+3)/q^8$
212121212	$-(q^2+1)(q^4-q^2+1)(q^4+1)/q^5$
121212121	$-(q^2+1)(q^4-q^2+1)(q^4+1)/q^5$
1212121212	1
2121212121	1

Figure 6: Last twelve coefficients of u_0

σ	$Coefficient$
1	$1/2\,(q^4-q^2+1)^2(q^8-q^6+q^4-q^2+1)^2(q^4+1)^2(q^2+1)^4$ $\times(x+3\,q^{28}+4\,q^{24}-2\,q^{22}+10\,q^{20}-2\,q^{18}+4\,q^{16}+3\,q^{12})/q^{40}$
121	$-1/2\,(q^2+1)^2(2\,q^{18}-295\,q^{28}-516\,q^{36}+x+210\,q^{26}+3\,q^{56}-3\,q^{54}+47\,q^{46}+9\,q^{52}-q^{48}$ $+2\,q^{50}-84\,q^{24}-295\,q^{40}+604\,q^{34}+462\,q^{30}-xq^2+47\,q^{22}-9\,q^{20}x+19\,q^{10}x-q^{26}x+q^{28}x-3\,q^{14}$ $+q^{24}x+4\,q^{22}x+30\,q^{14}x+462\,q^{38}-516\,q^{32}+19\,q^{18}x+210\,q^{42}-q^{20}+9\,q^{16}-24\,q^{16}x-24\,q^{12}x$ $-9\,q^8x+4\,q^6x+q^4x-84\,q^{44}+3\,q^{12})/q^{36}$
12121	$1/2\,(q^{18}-2\,q^{28}+22\,q^{36}+x+45\,q^{26}+2\,q^{46}+3\,q^{48}+22\,q^{24}+24\,q^{40}+45\,q^{34}+92\,q^{30}+18\,q^{22}+q^{20}x$ $+6\,q^{10}x+2\,q^{14}+q^{14}x+18\,q^{38}-2\,q^{32}+q^{42}+24\,q^{20}+9\,q^{16}+q^{16}x+q^6x+q^4x+9\,q^{44}+3\,q^{12})/q^{30}$
1212121	$-1/2\,(22\,q^{20}+6\,q^{16}+6\,q^{24}+2\,q^{26}+2\,q^{14}+2\,q^{30}+2\,q^{10}+3\,q^{28}-2\,q^{22}-2\,q^{18}+3\,q^{12}+x)/q^{20}$
121212121	1

Figure 7: Coefficients of u_1^1

σ	$Coefficient$
1	$(q^8 - q^6 + q^4 - q^2 + 1)^2 (q^{12} + q^{10} + 2\,q^8 + 2\,q^4 + q^2 + 1)^2 (q^2 + 1)^4 (q^4 - q^2 + 1)^4 / q^{32}$
121	$-(1 - 4\,q^{10} + 14\,q^8 - 30\,q^{14} + 44\,q^{28} + 73\,q^{16} + 3\,q^2 + 14\,q^{32} - 30\,q^{26} + 73\,q^{24} + 3\,q^{38}$ $+102\,q^{20} - 53\,q^{18} + q^{40} + 44\,q^{12} - 53\,q^{22} - 4\,q^{30} + 5\,q^4 + 5\,q^{36})(q^2 + 1)^2 (q^4 - q^2 + 1)^2 / q^{26}$
12121	$(3 + 72\,q^{18} + 14\,q^{28} + 3\,q^{36} + 20\,q^{26} + 10\,q^{24} + 2\,q^{34} + 2\,q^{30} + 36\,q^{22} + 14\,q^8 + 7\,q^4$ $+2\,q^2 + 7\,q^{32} + 2\,q^6 - 10\,q^{20} - 10\,q^{16} + 10\,q^{12} + 20\,q^{10} + 36\,q^{14}) / q^{18}$
1212121	$-(1 - 2\,q^{12} + 14\,q^{10} - 2\,q^8 + q^4 + 3\,q^2 + 4\,q^6 + q^{20} + 4\,q^{14} + 3\,q^{18} + q^{16}) / q^{10}$
121212121	1

Figure 8: Coefficients of u_3^1

σ	$Coefficient$
1	$(q^2+1)^2(q^4-q^2+1)^2(q^4+1)^2(q^8-q^6+q^4-q^2+1)^2(q^{12}+q^{10}+2\,q^8+2\,q^4+q^2+1)^2/q^{30}$
121	$-(1+75\,q^{18}-49\,q^{28}+42\,q^{36}+206\,q^{26}-q^{46}+q^{52}+7\,q^{48}-49\,q^{24}+40\,q^{40}+75\,q^{34}+158\,q^{30}+158\,q^{22}$ $+22\,q^8+7\,q^4+17\,q^{14}+17\,q^{38}+q^{32}-q^6+q^{20}+42\,q^{16}+22\,q^{44}+40\,q^{12})q^{26}$
12121	$(3+80\,q^{18}+10\,q^{28}+3\,q^{36}+26\,q^{26}-3\,q^{24}+q^{34}+5\,q^{30}+52\,q^{22}+10\,q^8+5\,q^4+52\,q^{14}+q^2+5\,q^{32}$ $+5\,q^6-19\,q^{20}-19\,q^{16}-3\,q^{12}+26\,q^{10})/q^{18}$
1212121	$-(1-2\,q^{12}+14\,q^{10}-2\,q^8+3\,q^2+5\,q^6+q^{20}+5\,q^{14}+3\,q^{18})/q^{10}$
121212121	1

Figure 9: Coefficients of u_4^1

σ	$Coefficient$
1	$(q^2+1)^2(q^4+1)^2(q^{12}+q^{10}+2\,q^8+2\,q^4+q^2+1)^2(q^4-q^2+1)^4/q^{26}$
121	$-(q^{36}+3\,q^{34}+10\,q^{32}+19\,q^{30}+33\,q^{28}+53\,q^{26}+64\,q^{24}+91\,q^{22}+84\,q^{20}+116\,q^{18}+84\,q^{16}+91\,q^{14}$ $+64\,q^{12}+53\,q^{10}+33\,q^8+19\,q^6+10\,q^4+3\,q^2+1)(q^4-q^2+1)^2/q^{22}$
12121	$(80\,q^{16}+3\,q^{26}+26\,q^{24}+4\,q^{22}+q^{32}+7\,q^{28}+50\,q^{20}+3\,q^6+3\,q^2+50\,q^{12}+1+3\,q^{30}+7\,q^4-14\,q^{18}$ $-14\,q^{14}+4\,q^{10}+26\,q^8)/q^{16}$
1212121	$-(3+5\,q^{12}-2\,q^{10}+14\,q^8+5\,q^4+q^2-2\,q^6+q^{14}+3\,q^{16})/q^8$
121212121	1

Figure 10: Coefficients of u_5^1

g_1	$-1/2\,\frac{-3\,q^{28}-4\,q^{24}+2\,q^{22}-10\,q^{20}+2\,q^{18}-4\,q^{16}-3\,q^{12}+x}{q^{20}}$

g_3	$\frac{\left(q^4+1\right)^2}{q^4}$

g_4	$\frac{\left(q^2+1\right)^2\left(q^4-q^2+1\right)^2}{q^6}$

g_5	$\frac{\left(q^2+1\right)^2\left(q^8-q^6+q^4-q^2+1\right)^2}{q^{10}}$

Figure 11: The elements g_i

dozen such relations. Because of the nature of these generating relations, there is no good "standard monomial basis" for $\mathcal{B}$ as for the Hecke algebra or for the $r = 3$ case in Section 7.1.1. Fortunately, this makes no difference as far as duality and reciprocity is concerned, as we shall see here, and also as far as existence of a canonical basis is concerned, as we shall see in [GCT8].

It was verified by computer that $\mathcal{B}$ is of dimension 114 [GCT4]. Since it is semisimple, it admits a Wederburn structure decomposition. It turns out that a complete Wederburn structure decomposition of the form (13) works over $Q(q)$ itself; i.e., no algebraic extension of $Q(q)$ is necessary here, just as in the case of Hecke algebras. This may be conjectured to be the case for the Kronecker problem in general, though it is not so for the plethysm problem in general as we already saw in Section 7.1.

So let

$$\mathcal{B} = \otimes_i T_{i,L} \otimes T_{i,R}, \tag{38}$$

be the complete Wederburn structure decomposition of $\mathcal{B}$, where $T_i = T_{i,L}$ ranges over all irreducible left $\mathcal{B}$-modules.

7.2.1 Irreducible representations

We describe these T_i next. There are two distinct irreducible representations of $\mathcal{B}$ of dimension $1, 2, 3$ and 5 each, and one of dimension 6. Since

$$114 = 1^2 + 1^2 + 2^2 + 2^2 + 3^2 + 3^2 + 5^2 + 5^2 + 6^2,$$

this is consistent with the Wederburn structure decomposition in (38).

Let $S_{q,\lambda}$ denote the q-Specht module of the Hecke algebra $\mathcal{H}_r(q)$ for the partition λ, and KL_λ its Kazhdan-Lusztig basis ordered appropriately.

σ	$Coefficient$
1	$1/2\,(q^2+1)^5(q^4-q^2+1)^3(q^8-q^6+q^4-q^2+1)^2(q^4+1)^3(-3\,q^{28}-4\,q^{24}+2\,q^{22}$ $-10\,q^{20}+2\,q^{18}-4\,q^{16}-3\,q^{12}+x)/q^{45}$
12	$-1/2\,(q^2+1)^4(q^4-q^2+1)^2(q^8-q^6+q^4-q^2+1)^2(q^4+1)^2(-3\,q^{28}-4\,q^{24}+2\,q^{22}$ $-10\,q^{20}+2\,q^{18}-4\,q^{16}-3\,q^{12}+x)/q^{40}$
121	$-1/2\,(q^2+1)^3(x+516\,q^{32}-462\,q^{38}-47\,q^{22}+q^{20}+84\,q^{24}+q^{48}-210\,q^{42}+84\,q^{44}-47\,q^{46}-210\,q^{26}$ $-2\,q^{18}-9\,q^{16}+3\,q^{14}+xq^{28}-xq^{26}+4\,xq^{22}+xq^{24}-24\,xq^{12}+19\,xq^{10}+30\,xq^{14}-9\,xq^{8}-24\,xq^{16}$ $+4\,xq^{6}-9\,xq^{20}+xq^{4}-xq^{2}+19\,xq^{18}+295\,q^{28}+295\,q^{40}-604\,q^{34}+516\,q^{36}-2\,q^{50}+3\,q^{54}-3\,q^{56}$ $-9\,q^{52}-462\,q^{30}-3\,q^{12})(q^4+1)(q^4-q^2+1)/q^{41}$
1212	$1/2\,(q^2+1)^2(x+516\,q^{32}-462\,q^{38}-47\,q^{22}+q^{20}+84\,q^{24}+q^{48}-210\,q^{42}+84\,q^{44}-47\,q^{46}-210\,q^{26}$ $-2\,q^{18}-9\,q^{16}+3\,q^{14}+xq^{28}-xq^{26}+4\,xq^{22}+xq^{24}-24\,xq^{12}+19\,xq^{10}+30\,xq^{14}-9\,xq^{8}-24\,xq^{16}$ $+4\,xq^{6}-9\,xq^{20}+xq^{4}-xq^{2}+19\,xq^{18}+295\,q^{28}+295\,q^{40}-604\,q^{34}+516\,q^{36}-2\,q^{50}+3\,q^{54}-3\,q^{56}$ $-9\,q^{52}-462\,q^{30}-3\,q^{12})/q^{36}$
12121	$1/2\,(x+2\,q^{32}-18\,q^{38}-18\,q^{22}-24\,q^{20}-22\,q^{24}-3\,q^{48}-q^{42}-9\,q^{44}-2\,q^{46}-45\,q^{26}-q^{18}-9\,q^{16}$ $-2\,q^{14}+6\,xq^{10}+xq^{14}+xq^{16}+xq^{6}+xq^{20}+xq^{4}+2\,q^{28}-24\,q^{40}-45\,q^{34}-22\,q^{36}-92\,q^{30}-3\,q^{12})$ $\times(q^2+1)(q^4+1)(q^4-q^2+1)/q^{35}$
121212	$-1/2\,(x+2\,q^{32}-18\,q^{38}-18\,q^{22}-24\,q^{20}-22\,q^{24}-3\,q^{48}-q^{42}-9\,q^{44}-2\,q^{46}-45\,q^{26}-q^{18}-9\,q^{16}$ $-2\,q^{14}+6\,xq^{10}+xq^{14}+xq^{16}+xq^{6}+xq^{20}+xq^{4}+2\,q^{28}-24\,q^{40}-45\,q^{34}-22\,q^{36}-92\,q^{30}-3\,q^{12})/q^{30}$
1212121	$-1/2\,(-2\,q^{10}-22\,q^{20}+2\,q^{18}-2\,q^{14}-3\,q^{12}-6\,q^{16}-2\,q^{30}+2\,q^{22}-6\,q^{24}-3\,q^{28}-2\,q^{26}+x)(q^2+1)$ $\times(q^4+1)(q^4-q^2+1)/q^{25}$
12121212	$1/2\,(-2\,q^{10}-22\,q^{20}+2\,q^{18}-2\,q^{14}-3\,q^{12}-6\,q^{16}-2\,q^{30}+2\,q^{22}-6\,q^{24}-3\,q^{28}-2\,q^{26}+x)/q^{20}$
121212121	$-(q^4+1)(q^4-q^2+1)(q^2+1)/q^5$
1212121212	1

Figure 12: Nonzero coefficients of z_0

$Monomial$	$Coefficient$
$x_2 \otimes x_1 \otimes x_0$	$-1/2\,(q^4+1)^2(q^2-q+1)(q^2+q+1)(q^8+1)^2(5\,x+173\,q^{32}-192\,q^{38}-192\,q^{22}+91\,q^{20}+117\,q^{24}+11\,q^{48}-45\,q^{42}+24\,q^{44}-37\,q^{46}-131\,q^{26}-45\,q^{18}+24\,q^{16}-37\,q^{14}+19\,xq^{12}-40\,xq^{10}-7\,xq^{14}+19\,xq^8+8\,xq^{16}-7\,xq^6+5\,xq^{20}+8\,xq^4-17\,xq^2-17\,xq^{18}+173\,q^{28}+91\,q^{40}-131\,q^{34}+117\,q^{36}-310\,q^{30}+11\,q^{12})/q^{51}$
$x_1 \otimes x_2 \otimes x_0$	$1/2\,(q^2+1)(q^2+q+1)(q^2-q+1)(q^4+1)(q^8+1)^2(5\,x+173\,q^{32}-192\,q^{38}-192\,q^{22}+91\,q^{20}+117\,q^{24}+11\,q^{48}-45\,q^{42}+24\,q^{44}-37\,q^{46}-131\,q^{26}-45\,q^{18}+24\,q^{16}-37\,q^{14}+19\,xq^{12}-40\,xq^{10}-7\,xq^{14}+19\,xq^8+8\,xq^{16}-7\,xq^6+5\,xq^{20}+8\,xq^4-17\,xq^2-17\,xq^{18}+173\,q^{28}+91\,q^{40}-131\,q^{34}+117\,q^{36}-310\,q^{30}+11\,q^{12})/q^{46}$
$x_0 \otimes x_3 \otimes x_0$	$-1/2\,(q+1)(q-1)(q^4+1)(q^2+q+1)^2(q^2-q+1)^2(q^8+1)^2(3\,x+727\,q^{32}-500\,q^{38}-330\,q^{22}+191\,q^{20}+460\,q^{24}+59\,q^{48}-359\,q^{42}+192\,q^{44}-110\,q^{46}-603\,q^{26}-138\,q^{18}+76\,q^{16}-35\,q^{14}-20\,xq^{22}+5\,xq^{24}+70\,xq^{12}-58\,xq^{10}-77\,xq^{14}+45\,xq^8+48\,xq^{16}-46\,xq^6+23\,xq^{20}+30\,xq^4-15\,xq^2-32\,xq^{18}+587\,q^{28}+402\,q^{40}-780\,q^{34}+584\,q^{36}-44\,q^{50}+11\,q^{52}-685\,q^{30}+7\,q^{12})/q^{53}$
$x_2 \otimes x_0 \otimes x_1$	$1/2\,(q^4+1)^2(q^2-q+1)(q^2+q+1)(q^8+1)^2(5\,x+173\,q^{32}-192\,q^{38}-192\,q^{22}+91\,q^{20}+117\,q^{24}+11\,q^{48}-45\,q^{42}+24\,q^{44}-37\,q^{46}-131\,q^{26}-45\,q^{18}+24\,q^{16}-37\,q^{14}+19\,xq^{12}-40\,xq^{10}-7\,xq^{14}+19\,xq^8+8\,xq^{16}-7\,xq^6+5\,xq^{20}+8\,xq^4-17\,xq^2-17\,xq^{18}+173\,q^{28}+91\,q^{40}-131\,q^{34}+117\,q^{36}-310\,q^{30}+11\,q^{12})/q^{48}$
$x_1 \otimes x_1 \otimes x_1$	$-1/2\,(q^2+1)^2(q^4+1)(q-1)(q+1)(q^2+q+1)(q^2-q+1)(q^8+1)^2(5\,x+173\,q^{32}-192\,q^{38}-192\,q^{22}+91\,q^{20}+117\,q^{24}+11\,q^{48}-45\,q^{42}+24\,q^{44}-37\,q^{46}-131\,q^{26}-45\,q^{18}+24\,q^{16}-37\,q^{14}+19\,xq^{12}-40\,xq^{10}-7\,xq^{14}+19\,xq^8+8\,xq^{16}-7\,xq^6+5\,xq^{20}+8\,xq^4-17\,xq^2-17\,xq^{18}+173\,q^{28}+91\,q^{40}-131\,q^{34}+117\,q^{36}-310\,q^{30}+11\,q^{12})/q^{47}$
$x_0 \otimes x_2 \otimes x_1$	$1/2\,(q^4+1)(q^2+q+1)(q^2-q+1)(q^8+1)^2(-3\,x-1316\,q^{32}+1364\,q^{38}+419\,q^{22}-200\,q^{20}-577\,q^{24}-464\,q^{48}+957\,q^{42}-617\,q^{44}+613\,q^{46}+748\,q^{26}+5\,q^{30}x+134\,q^{18}-76\,q^{16}+35\,q^{14}-25\,xq^{28}+35\,xq^{26}+67\,xq^{22}-28\,xq^{24}-117\,xq^{12}+97\,xq^{10}+110\,xq^{14}-48\,xq^8-85\,xq^{16}+44\,xq^6-112\,xq^{20}-30\,xq^4+15\,xq^2+123\,xq^{18}-816\,q^{28}-1224\,q^{40}+1325\,q^{34}-1132\,q^{36}+257\,q^{50}+85\,q^{54}-55\,q^{56}+11\,q^{58}-108\,q^{52}+1220\,q^{30}-7\,q^{12})/q^{50}$

Figure 13: First five nonzero coefficients of $a \in X_q^{\otimes 3}$

Monomial	*Coefficient*
$x_0 \otimes x_0 \otimes x_2$	$-1/2\,(q^2+1)(q^2+q+1)(q^2-q+1)(q^4+1)(q^8+1)^2(5\,x+173\,q^{32}-192\,q^{38}-192\,q^{22}+91\,q^{20}+117\,q^{24}$ $+11\,q^{48}-45\,q^{42}+24\,q^{44}-37\,q^{46}-131\,q^{26}-45\,q^{18}+24\,q^{16}-37\,q^{14}+19\,xq^{12}-40\,xq^{10}-7\,xq^{14}$ $+19\,xq^{8}+8\,xq^{16}-7\,xq^{6}+5\,xq^{20}+8\,xq^{4}-17\,xq^{2}-17\,xq^{18}+173\,q^{28}+91\,q^{40}-131\,q^{34}+117\,q^{36}$ $-310\,q^{30}+11\,q^{12})/q^{46}$
$x_0 \otimes x_1 \otimes x_2$	$1/2\,(q^4+1)(q^2+q+1)(q^2-q+1)(q^8+1)^2(3\,x+951\,q^{32}-1060\,q^{38}-363\,q^{22}+176\,q^{20}+449\,q^{24}+248\,q^{48}$ $-592\,q^{42}+395\,q^{44}-451\,q^{46}-532\,q^{26}-97\,q^{18}+65\,q^{16}-35\,q^{14}+8\,xq^{28}-27\,xq^{26}-31\,xq^{22}+16\,xq^{24}$ $+81\,xq^{12}-85\,xq^{10}-62\,xq^{14}+40\,xq^{8}+59\,xq^{16}-27\,xq^{6}+64\,xq^{20}+25\,xq^{4}-15\,xq^{2}-97\,xq^{18}+654\,q^{28}$ $+797\,q^{40}-898\,q^{34}+828\,q^{36}-129\,q^{50}-61\,q^{54}+18\,q^{56}+52\,q^{52}-998\,q^{30}+7\,q^{12})/q^{49}$
$x_0 \otimes x_0 \otimes x_3$	$-1/2\,(q^4+1)(q-1)^2(q+1)^2(q^8+1)^2(q^2+q+1)^3(q^2-q+1)^3(3\,x+275\,q^{32}-94\,q^{38}-220\,q^{22}$ $+132\,q^{20}+275\,q^{24}-35\,q^{42}+7\,q^{44}-279\,q^{26}-94\,q^{18}+65\,q^{16}-35\,q^{14}+25\,xq^{12}-26\,xq^{10}-15\,xq^{14}$ $+22\,xq^{8}+3\,xq^{16}-26\,xq^{6}+25\,xq^{4}-15\,xq^{2}+250\,q^{28}+65\,q^{40}-220\,q^{34}+132\,q^{36}-279\,q^{30}+7\,q^{12})/q^{50}$

Figure 14: Last four nonzero coefficients of a

Monomial	*Coefficient*
$x_3 \otimes x_0 \otimes x_0$	$1/2\,(q^4+1)(q-1)^2(q+1)^2(q^8+1)^2(q^2+q+1)^3(q^2-q+1)^3(5\,x+1214\,q^{32}-847\,q^{38}-525\,q^{22}$ $+289\,q^{20}+714\,q^{24}+107\,q^{48}-525\,q^{42}+289\,q^{44}-178\,q^{46}-847\,q^{26}-178\,q^{18}+107\,q^{16}-55\,q^{14}-25\,xq^{22}$ $+5\,xq^{24}+130\,xq^{12}-113\,xq^{10}-113\,xq^{14}+75\,xq^{8}+75\,xq^{16}-60\,xq^{6}+45\,xq^{20}+45\,xq^{4}-25\,xq^{2}-60\,xq^{18}$ $+920\,q^{28}+714\,q^{40}-1139\,q^{34}+920\,q^{36}-55\,q^{50}+11\,q^{52}-1139\,q^{30}+11\,q^{12})/q^{57}$
$x_2 \otimes x_1 \otimes x_0$	$-1/2\,(q^4+1)(q-1)^2(q+1)^2(q^2+q+1)^2(q^2-q+1)^2(q^8+1)^2(-5\,x-1170\,q^{32}+977\,q^{38}+425\,q^{22}$ $-207\,q^{20}-565\,q^{24}-371\,q^{48}+636\,q^{42}-451\,q^{44}+397\,q^{46}+523\,q^{26}+5\,q^{30}x+106\,q^{18}-89\,q^{16}+55\,q^{14}$ $-20\,xq^{28}+20\,xq^{26}+43\,xq^{22}-20\,xq^{24}-119\,xq^{12}+97\,xq^{10}+71\,xq^{14}-45\,xq^{8}-59\,xq^{16}+28\,xq^{6}-95\,xq^{20}$ $-37\,xq^{4}+25\,xq^{2}+87\,xq^{18}-676\,q^{28}-1021\,q^{40}+891\,q^{34}-849\,q^{36}+173\,q^{50}+52\,q^{54}-44\,q^{56}+11\,q^{58}$ $-82\,q^{52}+1002\,q^{30}-11\,q^{12})/q^{56}$
$x_1 \otimes x_2 \otimes x_0$	$1/2\,(q^4+1)(q-1)^2(q+1)^2(q^2+q+1)^2(q^2-q+1)^2(q^8+1)^2(-8\,x-1447\,q^{32}+1598\,q^{38}+590\,q^{22}$ $-304\,q^{20}-648\,q^{24}-496\,q^{48}+927\,q^{42}-742\,q^{44}+696\,q^{46}+778\,q^{26}+5\,q^{30}x+153\,q^{18}-89\,q^{16}+72\,q^{14}$ $-25\,xq^{28}+40\,xq^{26}+58\,xq^{22}-40\,xq^{24}-124\,xq^{12}+138\,xq^{10}+96\,xq^{14}-76\,xq^{8}-114\,xq^{16}+39\,xq^{6}-116\,xq^{20}$ $-33\,xq^{4}+32\,xq^{2}+152\,xq^{18}-1064\,q^{28}-1329\,q^{40}+1319\,q^{34}-1386\,q^{36}+244\,q^{50}+96\,q^{54}-55\,q^{56}$ $+11\,q^{58}-134\,q^{52}+1516\,q^{30}-18\,q^{12})/q^{55}$
$x_0 \otimes x_3 \otimes x_0$	$1/2\,(q^4+1)(q-1)^2(q+1)^2(q^8+1)^2(q^2-q+1)^3(q^2+q+1)^3(-3\,x-1693\,q^{32}+1146\,q^{38}+641\,q^{22}-404\,q^{20}$ $-1020\,q^{24}-178\,q^{48}+814\,q^{42}-567\,q^{44}+296\,q^{46}+1268\,q^{26}+266\,q^{18}-155\,q^{16}+53\,q^{14}+5\,xq^{26}+45\,xq^{22}$ $-25\,xq^{24}-164\,xq^{12}+123\,xq^{10}+170\,xq^{14}-108\,xq^{8}-131\,xq^{16}+96\,xq^{6}-60\,xq^{20}-65\,xq^{4}+23\,xq^{2}$ $+78\,xq^{18}-1376\,q^{28}-1006\,q^{40}+1709\,q^{34}-1491\,q^{36}+107\,q^{50}+11\,q^{54}-55\,q^{52}+1449\,q^{30}-7\,q^{12})/q^{58}$
$x_2 \otimes x_0 \otimes x_1$	$-1/2\,(q^2+1)(q^4+1)(q-1)^2(q+1)^2(q^2-q+1)^2(q^2+q+1)^2(q^8+1)^2(5\,x+964\,q^{32}-627\,q^{38}$ $-431\,q^{22}+224\,q^{20}+582\,q^{24}+100\,q^{48}-431\,q^{42}+224\,q^{44}-143\,q^{46}-627\,q^{26}-143\,q^{18}+100\,q^{16}-55\,q^{14}-25\,xq^{22}$ $+5\,xq^{24}+108\,xq^{12}-87\,xq^{10}-87\,xq^{14}+50\,xq^{8}+50\,xq^{16}-45\,xq^{6}+42\,xq^{20}+42\,xq^{4}-25\,xq^{2}-45\,xq^{18}+645\,q^{28}$ $+582\,q^{40}-860\,q^{34}+645\,q^{36}-55\,q^{50}+11\,q^{52}-860\,q^{30}+11\,q^{12})/q^{53}$

Figure 15: First five nonzero coefficients of b

$Monomial$	$Coefficient$
$x_1 \otimes x_1 \otimes x_1$	$1/2\,(q^2+1)^2(q^2-q+1)(q^2+q+1)(q^4+1)(q-1)^2(q+1)^2(q^8+1)^2(3\,x+277\,q^{32}-621\,q^{38}-165\,q^{22} +97\,q^{20}+83\,q^{24}+125\,q^{48}-291\,q^{42}+291\,q^{44}-299\,q^{46}-255\,q^{26}-47\,q^{18}-17\,q^{14}+5\,xq^{28}-20\,xq^{26}-15\,xq^{22} +20\,xq^{24}+5\,xq^{12}-41\,xq^{10}-25\,xq^{14}+31\,xq^{8}+55\,xq^{16}-11\,xq^{6}+21\,xq^{20}-4\,xq^{4}-7\,xq^{2}-65\,xq^{18}+388\,q^{28} +308\,q^{40}-428\,q^{34}+537\,q^{36}-71\,q^{50}-44\,q^{54}+11\,q^{56}+52\,q^{52}-514\,q^{30}+7\,q^{12})/q^{52}$
$x_0 \otimes x_2 \otimes x_1$	$-1/2\,(q^2+1)(q^4+1)(q-1)^2(q+1)^2(q^2-q+1)^2(q^2+q+1)^2(q^8+1)^2(-3\,x-1651\,q^{32}+1803\,q^{38} +486\,q^{22}-287\,q^{20}-679\,q^{24}-567\,q^{48}+1181\,q^{42}-1013\,q^{44}+814\,q^{46}+920\,q^{26}+5\,q^{30}x+147\,q^{18}-72\,q^{16} +35\,q^{14}-25\,xq^{28}+45\,xq^{26}+78\,xq^{22}-60\,xq^{24}-133\,xq^{12}+122\,xq^{10}+138\,xq^{14}-87\,xq^{8}-167\,xq^{16} +49\,xq^{6}-131\,xq^{20}-28\,xq^{4}+15\,xq^{2}+170\,xq^{18}-1270\,q^{28}-1556\,q^{40}+1669\,q^{34}-1825\,q^{36}+296\,q^{50} +107\,q^{54}-55\,q^{56}+11\,q^{58}-178\,q^{52}+1547\,q^{30}-7\,q^{12})/q^{55}$
$x_1 \otimes x_0 \otimes x_2$	$-1/2\,(q^4+1)^2(q-1)^2(q+1)^2(q^8+1)^2(q^2-q+1)^3(q^2+q+1)^3(3\,x+275\,q^{32}-94\,q^{38}-220\,q^{22} +132\,q^{20}+275\,q^{24}-35\,q^{42}+7\,q^{44}-279\,q^{26}-94\,q^{18}+65\,q^{16}-35\,q^{14}+25\,xq^{12}-26\,xq^{10} -15\,xq^{14}+22\,xq^{8}+3\,xq^{16}-26\,xq^{6}+25\,xq^{4}-15\,xq^{2}+250\,q^{28}+65\,q^{40}-220\,q^{34}+132\,q^{36}-279\,q^{30}+7\,q^{12})/q^{51}$
$x_0 \otimes x_1 \otimes x_2$	$-1/2\,(q^4+1)^2(q-1)^2(q+1)^2(q^8+1)^2(q^2-q+1)^3(q^2+q+1)^3(3\,x+275\,q^{32}-94\,q^{38}-220\,q^{22} +132\,q^{20}+275\,q^{24}-35\,q^{42}+7\,q^{44}-279\,q^{26}-94\,q^{18}+65\,q^{16}-35\,q^{14}+25\,xq^{12}-26\,xq^{10}-15\,xq^{14}+22\,xq^{8} +3\,xq^{16}-26\,xq^{6}+25\,xq^{4}-15\,xq^{2}+250\,q^{28}+65\,q^{40}-220\,q^{34}+132\,q^{36}-279\,q^{30}+7\,q^{12})/q^{54}$
$x_0 \otimes x_0 \otimes x_3$	$1/2\,(q^2+1)(q^4+1)^2(q^4-q^2+1)(q-1)^2(q+1)^2(q^8+1)^2(q^2-q+1)^3(q^2+q+1)^3(3\,x+275\,q^{32} -94\,q^{38}-220\,q^{22}+132\,q^{20}+275\,q^{24}-35\,q^{42}+7\,q^{44}-279\,q^{26}-94\,q^{18}+65\,q^{16}-35\,q^{14}+25\,xq^{12}-26\,xq^{10}-15\,xq^{14} +22\,xq^{8}+3\,xq^{16}-26\,xq^{6}+25\,xq^{4}-15\,xq^{2}+250\,q^{28}+65\,q^{40}-220\,q^{34}+132\,q^{36}-279\,q^{30}+7\,q^{12})/q^{55}$

Figure 16: Last five nonzero coefficients of b

Since, in this case, $\mathcal{B} = \mathcal{B}_r^H(q) \subseteq \mathcal{H}_r(q) \otimes \mathcal{H}_r(q)$, the tensor product $S_{q,\lambda} \otimes S_{q,\mu}$ is a representation of $\mathcal{B}$. In particular,

$$T_{q,\lambda} = S_{q,\lambda} \otimes S_{q,(r)} \cong S_{q,(r)} \otimes S_{q,\lambda},$$

where $S_{q,(r)}$ is the trivial one dimensional q-Specht module, is an irreducible $\mathcal{B}$-module, which specializes at $q = 1$ to the Specht module S_λ of the symmetric group S_r.

Let

$$\begin{array}{rcl} T_0 & = & T_{q,(4)}, \\ T_1 & = & T_{q,(1,1,1,1)}, \\ T_2 & = & T_{q,(2,2)}, \\ T_3 & = & T_{q,(2,1,1)}, \\ T_4 & = & T_{q,(3,1)}, \\ T_5 & = & S_{q,(3,1)} \otimes S_{q,(2,2)} \cong S_{q,(2,1,1)} \otimes S_{q,(2,2)}. \end{array} \tag{39}$$

These are irreducible $\mathcal{B}$-modules. Their dimensions are $1, 1, 2, 3, 3$ and 6 respectively.

To get the other two dimensional irreducible $\mathcal{B}$-module, we analyze how the tensor product $S_{q,(2,2)} \otimes S_{q,(2,2)}$ decomposes as a $\mathcal{B}$-module. It decomposes as:

$$S_{q,(2,2)} \otimes S_{q,(2,2)} \cong T_{q,(4)} \oplus T_{q,(1,1,1,1)} \oplus T_6,$$

where T_6 is the other two dimensional irreducible $\mathcal{B}$-module that we were looking for. Explicitly, a basis of T_6 in terms of the Kazhdan-Lusztig basis $KL_{(2,2)} \otimes KL_{(2,2)}$ of $S_{q,(2,2)} \otimes S_{q,(2,2)}$ is given by the rows of the matrix

$$\begin{bmatrix} 1 & \frac{1+q}{2q^{1/2}} & \frac{1+q}{2q^{1/2}} & 0 \\ 0 & \frac{1+q}{2q^{1/2}} & \frac{1+q}{2q^{1/2}} & 1 \end{bmatrix}.$$

Matrix representations of the right action of the generators $\mathcal{Q}_i$'s of $\mathcal{B}$ on this basis are:

$$\mathcal{Q}_1 = \mathcal{Q}_3 = \begin{bmatrix} (1+q)^2/q & 0 \\ (1+q^2)/q & 0 \end{bmatrix}$$

$$\mathcal{Q}_2 = \begin{bmatrix} 0 & (1+q^2)/q \\ 0 & (1+q)^2/q \end{bmatrix}$$

The specialization of T_6 at $q = 1$ is isomorphic to the Specht module $S_{(2,2)}$ of S_4. But T_6 is nonisomorphic to T_2, whose specialization at $q = 1$ is the same.

To get the five dimensional irreducible $\mathcal{B}$-modules, we analyze how the tensor products $S_{q,(2,1,1)} \otimes S_{q,(2,1,1)}$ and $S_{q,(3,1)} \otimes S_{q,(2,1,1)}$ decompose as $\mathcal{B}$-modules. We have

$$S_{q,(2,1,1)} \otimes S_{q,(2,1,1)} \cong T_{q,(2,1,1)} \oplus T_{q,(4)} \oplus T_7,$$

where T_7 is the first five dimensional irreducible $\mathcal{B}$-representation that we were looking for. Explicitly, its basis in terms of the Kazhdan-Lusztig basis $KL_{(2,1,1)} \otimes KL_{(2,1,1)}$ is given by the rows of the matrix:

$$\begin{array}{llllllll}
w_1 = [& 0 & -(1+q)/(2q^{1/2}) & -(1+q)^2/(2q) & 0 & 0 & 0] \\
w_2 = [& -1 & -(1+q)/(2q^{1/2}) & 0 & 0 & (1+q)/(2q^{1/2}) & 1] \\
w_3 = [& 0 & 0 & -(1+q)^2/(2q) & 0 & -(1+q)/(2q^{1/2}) & 0] \\
v_1 = [& 0 & -(1+q)/(2q^{1/2}) & -(1+q)^2/(2q) & -1 & -(1+q)/(2q^{1/2}) & 0] \\
v_2 = [& 1 & (1+q)/(2q^{1/2}) & 1 & 0 & (1+q)/(2q^{1/2}) & 1]
\end{array}$$

Matrix representations of the right action of $\mathcal{Q}_i$'s in this basis are:

$$\mathcal{Q}_1 = \begin{bmatrix} (1+q)^2/q & 0 & 0 & 0 & 0 \\ (1+q^2)/q & 0 & 0 & -(1+q^2)/q & 0 \\ 0 & 0 & 0 & (1+q)^2/q & 0 \\ 0 & 0 & 0 & (1+q)^2/q & 0 \\ (q-1)^2/q & 0 & 0 & -(1+q^2)/q & 0 \end{bmatrix}$$

$$\mathcal{Q}_2 = \begin{bmatrix} 0 & \frac{(1+q)^2}{2q} & 0 & 0 & -\frac{(1+q)^2}{2q} \\ 0 & \frac{(1+q)^2}{q} & 0 & 0 & 0 \\ 0 & -\frac{(1+q)^2}{2q} & 0 & 0 & -\frac{(1+q)^2}{2q} \\ 0 & 0 & 0 & 0 & -\frac{1+q^2}{q} \\ 0 & 0 & 0 & 0 & \frac{(1+q)^2}{q} \end{bmatrix}$$

$$\mathcal{Q}_3 = \begin{bmatrix} 0 & 0 & 0 & (1+q)^2/q & 0 \\ 0 & 0 & -(1+q^2)/q & (1+q^2)/q & 0 \\ 0 & 0 & (1+q)^2/q & 0 & 0 \\ 0 & 0 & 0 & (1+q)^2/q & 0 \\ 0 & 0 & (q-1)^2/q & -(1+q^2)/q & 0 \end{bmatrix}$$

Let V denote the span of the vectors v_1 and v_2, and $V(1)$ its specialization at $q = 1$. It can be checked that $V(1)$ is isomorphic to the Specht module

$S_{(2,2)}$ of S_4, and the quotient $T_7(1)/V$, where $T_7(1)$ denotes the specialization of T_7 at $q = 1$, is isomorphic to the Specht module $S_{(3,1)}$ of S_4.

Finally,

$$S_{q,(3,1)} \otimes S_{q,(2,1,1)} \cong T_{q,(3,1)} \oplus T_{q,(1,1,1,1)} \oplus T_8,$$

where $T_8 \not\cong T_7$ is the second five dimnsional irreducible $\mathcal{B}$-representation that we were looking for. Its basis and representation matrices are similar.

This specifies all irreducible representations of $\mathcal{B}$.

7.2.2 Duality

Using the explicit representations T_i above, the Wederburn structure decomposition (38) of $\mathcal{B}$ was explicitly determined with the help of a computer. The explicit bases of the structure components $U_i = T_{i,L} \otimes T_{i,R}$ in (38) are far too complex to be given here.

Fix any $u_i \in U_i$, $0 \leq i \leq 8$, and let $W_i = X_q^{\otimes r} \cdot u_i$ be the corresponding left representation of the nonstandard quantum group G_q^H. Computer experiments indicate that these are nonisomorphic irreducible representations of G_q^H with the following decompositions as H_q-modules, $H_q = GL_q(\mathbb{C}^2) \times GL_q(\mathbb{C}^2)$. (Recall that $V_{q,\lambda}(n)$ is the q-Weyl module of $GL_q(\mathbb{C}^n)$).

$$\begin{array}{rcl}
W_0 & \cong & V_{q,(4)}(2) \otimes V_{q,(4)}(2) \oplus V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2) \oplus V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2), \\
W_1 & \cong & V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2), \\
W_2 & \cong & V_{q,(4)}(2) \otimes V_{q,(2,2)}(2) \oplus V_{q,(2,2)}(2) \otimes V_{q,(4)}(2), \\
W_3 & \cong & V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2), \\
W_4 & \cong & V_{q,(3,1)}(2) \otimes V_{q,(4)}(2) \oplus V_{q,(4)}(2) \otimes V_{q,(3,1)}(2), \\
W_5 & \cong & V_{q,(2,2)}(2) \otimes V_{q,(3,1)}(2) \oplus V_{q,(3,1)}(2) \otimes V_{q,(2,2)}(2), \\
W_6 & \cong & V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2), \\
W_7 & \cong & V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2).
\end{array}$$

Their dimensions are $35, 1, 10, 9, 30, 6, 1$ and 9, respectively. The module W_8 turns out to be zero when $\dim(V) = \dim(W) = 2$, as here; however, it would be nonzero for general $\dim(V)$ and $\dim(W)$. Furthermore,

$$X_q^{\otimes 4} = \bigoplus_i W_i \otimes T_i,$$

in accordance with the duality conjecture.

Remark: These computations are not final. The main problem is that the symbolic computations needed here are too heavy for MATLAB/Maple to handle. Hence, in some of the computations q was set to a fixed real value (such as .5). This introduces floating point errors in various calculations. As far as we can see, this does not affect the decomposition above. But this has to be double checked by other means.

7.2.3 Reciprocity

Let m^i_μ denote the multiplicity of the Specht module S_μ of S_4 in T_i. Then it can be verified that

$$\begin{aligned}
&m^0_{(4)} = 1,\\
&m^1_{(1,1,1,1)} = 1,\\
&m^2_{(2,2)} = 1,\\
&m^3_{(2,1,1)} = 1,\\
&m^4_{(3,1)} = 1,\\
&m^5_{(3,1)} = m^5_{(2,1,1)} = 1,\\
&m^6_{(2,2)} = 1,\\
&m^7_{(3,1)} = m^7_{(2,2)} = 1,\\
&m^8_{(2,1,1)} = m^8_{(2,2)} = 1.
\end{aligned}$$

All other m^i_μ's are zero. It can now be seen that, as H_q-modules, $H_q =$

$GL_q(2) \times GL_q(2)$, we have

$$\begin{array}{lcl}
V_{q,(4)}(4) & \cong & m^0_{(4)} W_0 \\
& \cong & V_{q,(4)}(2) \otimes V_{q,(4)}(2) \oplus V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2) \\
& & \quad \oplus V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2), \\
& & \\
V_{q,(3,1)}(4) & \cong & m^4_{(3,1)} W_4 \oplus m^5_{(3,1)} W_5 \oplus m^7_{(3,1)} W_7 \\
& \cong & V_{q,(3,1)}(2) \otimes V_{q,(4)}(2) \oplus V_{q,(4)}(2) \otimes V_{q,(3,1)}(2) \\
& & \quad \oplus V_{q,(2,2)}(2) \otimes V_{q,(3,1)}(2) \oplus V_{q,(3,1)}(2) \otimes V_{q,(2,2)}(2) \\
& & \quad \oplus V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2), \\
& & \\
V_{q,(2,2)}(4) & \cong & m^2_{(2,2)} W_2 \oplus m^7_{(2,2)} W_7 \oplus m^6_{(2,2)} W_6 \\
& \cong & V_{q,(4)}(2) \otimes V_{q,(2,2)}(2) \oplus V_{q,(2,2)}(2) \otimes V_{q,(4)}(2) \\
& & \quad \oplus V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2) \oplus V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2), \\
& & \\
V_{q,(2,1,1)}(4) & \cong & m^3_{(2,1,1)} W_3 \oplus m^5_{(2,1,1)} W_5 \\
& \cong & V_{q,(3,1)}(2) \otimes V_{q,(3,1)}(2) \oplus V_{q,(2,2)}(2) \otimes V_{q,(3,1)}(2) \\
& & \quad \oplus V_{q,(3,1)}(2) \otimes V_{q,(2,2)}(2), \\
& & \\
V_{q,(1,1,1,1)}(4) & \cong & m^1_{(1,1,1,1)} W_1 \\
& \cong & V_{q,(2,2)}(2) \otimes V_{q,(2,2)}(2),
\end{array}$$

in accordance with the reciprocity conjecture.

We are unable to verify the refined reciprocity conjecture on computer since the necessary symbolic computations turn out to be beyond the reach of the desktop MATLAB/Maple.

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σ	$\hat{A}_\sigma$								
1	17738	17738	17550	17362	16994	16626	16114	15602	14933
	14264	13550	12836	12008	11180	10392	9604	8790	7976
	7226	6476	5806	5136	4518	3900	3418	2936	2504
	2072	1762	1452	1202	952	776	600	482	364
	280	196	152	108	77	46	34	22	14
	6	4	2	1					
121	20322	20322	20083	19844	19354	18864	18211	17558	16668
	15778	14890	14002	12934	11866	10916	9966	8962	7958
	7092	6226	5470	4714	4047	3380	2895	2410	1982
	1554	1287	1020	804	588	463	338	253	168
	122	76	53	30	19	8	5	2	1
12121	9078	9078	8973	8868	8623	8378	8051	7724	7245
	6766	6335	5904	5363	4822	4382	3942	3443	2944
	2562	2180	1851	1522	1267	1012	831	650	506
	362	284	206	151	96	70	44	28	12
	7	2	1						
1212121	1918	1918	1913	1908	1866	1824	1742	1660	1523
	1386	1277	1168	1042	916	821	726	603	480
	395	310	246	182	145	108	83	58	40
	22	14	6	3					
121212121	25032	24784	24124	23136	21978	20808	19710	18768	17934
	17160	16358	15440	14384	13168	11849	10484	9139	7880
	6725	5692	4765	3928	3170	2480	1868	1344	914
	584	345	188	93	40	15	4	1	

Figure 17: Positivity and unimodality of $\hat{A}_\sigma$'s

σ	$\hat{B}_\sigma$								
$\emptyset$	390464	389128	385120	378581	369652	358471	345176	330055	313396
	295506	276692	257207	237304	217316	197576	178283	159636	141795
	124920	109152	94632	81349	69292	58469	48888	40497	33244
	27011	21680	17198	13512	10505	8060	6095	4528	3314
	2408	1731	1204	812	540	357	232	147	84
	43	24	16	8	2				
2	102390	101996	100847	98976	96425	93236	89466	85172	80462
	75444	70190	64772	59280	53804	48429	43240	38271	33556
	29145	25088	21393	18068	15099	12472	10185	8236	6586
	5196	4040	3092	2333	1744	1286	920	640	440
	300	200	129	76	41	24	16	8	2
1	50420	50420	49799	49178	48066	46954	45325	43696	41665
	39634	37420	35206	32782	30358	27969	25580	23303	21026
	18902	16778	14947	13116	11553	9990	8713	7436	6455
	5474	4724	3974	3416	2858	2490	2122	1831	1540
	1350	1160	1031	902	779	656	582	508	441
	374	313	252	213	174	144	114	86	58
	47	36	27	18	11	4	4	4	2
12	13180	13086	12992	12744	12496	12124	11752	11225	10698
	10112	9526	8890	8254	7584	6914	6294	5674	5083
	4492	3979	3466	3036	2606	2256	1906	1638	1370
	1178	986	840	694	603	512	450	388	335
	282	259	236	206	176	153	130	116	102
	85	68	54	40	34	28	21	14	9
	4	4	4	2					

Figure 18: The vectors $\hat{B}_\sigma$

σ	$\hat{B}_\sigma$								
212	3432	3432	3379	3326	3242	3158	3033	2908	2744
	2580	2417	2254	2069	1884	1709	1534	1371	1208
	1062	916	797	678	581	484	411	338	287
	236	202	168	143	118	108	98	84	70
	65	60	56	52	43	34	30	26	23
	20	15	10	7	4	4	4	2	
121	51252	51252	50661	50070	48941	47812	46219	44626	42589
	40552	38328	36104	33645	31186	28756	26326	23948	21570
	19376	17182	15217	13252	11581	9910	8522	7134	6030
	4926	4106	3286	2664	2042	1632	1222	941	660
	497	334	233	132	89	46	25	4	-4
	-12	-10	-8	-6	-4	-4	-4	-2	
1212	13352	13285	13218	12957	12696	12341	11986	11462	10938
	10358	9778	9131	8484	7812	7140	6498	5856	5240
	4624	4088	3552	3079	2606	2236	1866	1552	1238
	1026	814	646	478	373	268	198	128	93
	58	35	12	4	-4	-4	-4	-4	-4
	-4	-4	-2						
21212	3472	3472	3427	3382	3293	3204	3093	2982	2810
	2638	2483	2328	2132	1936	1772	1608	1423	1238
	1098	958	820	682	587	492	398	304	249
	194	153	112	85	58	37	16	10	4
	2	0	-2	-4	-4	-4	-2		
12121	20922	20922	20625	20328	19815	19302	18558	17814	16848
	15882	14868	13854	12740	11626	10537	9448	8430	7412
	6509	5606	4830	4054	3439	2824	2349	1874	1526
	1178	945	712	550	388	301	214	155	96
	70	44	30	16	11	6	3		

Figure 19: The vectors $\hat{B}_\sigma$ (cont.)

σ	$\hat{B}_\sigma$								
121212	5496	5453	5410	5286	5162	5008	4854	4600	4346
	4068	3790	3497	3204	2894	2584	2295	2006	1749
	1492	1280	1068	889	710	586	462	366	270
	215	160	117	74	56	38	27	16	11
	6	3							
2121212	1434	1434	1406	1378	1346	1314	1267	1220	1128
	1036	961	886	799	712	633	554	470	386
	334	282	231	180	146	112	82	52	42
	32	24	16	11	6	3			
1212121	3800	3800	3735	3670	3573	3476	3326	3176	2974
	2772	2567	2362	2138	1914	1692	1470	1277	1084
	921	758	624	490	396	302	238	174	131
	88	65	42	29	16	11	6	3	
12121212	1004	992	980	957	934	908	882	829	776
	716	656	597	538	472	406	346	286	240
	194	158	122	94	66	51	36	26	16
	11	6	3						
212121212	258	258	252	246	245	244	237	230	208
	186	168	150	132	114	95	76	60	44
	37	30	23	16	11	6	3		
1212121212	68	67	66	65	64	63	62	58	54
	49	44	39	34	28	22	17	12	9
	6	4	2	1					

Figure 20: The vectors $\hat{B}_\sigma$ (cont)

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$Number$	$Coefficient$	σ
1	$\frac{20.q^0+104.q^1+256.q^2-113.q^3-49.q^4-113.q^5+256.q^6+104.q^7+20.q^8}{2.q^3+12.q^4+2.q^5}$	1
2	$\frac{-16.q^0-64.q^1-128.q^2-192.q^3-224.q^4-192.q^5-128.q^6-64.q^7-16.q^8}{2.q^3+12.q^4+2.q^5}$	2
3	$\frac{-4.q^0-16.q^1-28.q^2-32.q^3-28.q^4-16.q^5-4.q^6}{2.q^3}$	3
4	$\frac{1.q^0-4.q^2+6.q^4-4.q^6+1.q^8}{2.q^3+2.q^5}$	12
5	$\frac{-1.q^0-18.q^1-65.q^2-128.q^3-190.q^4-220.q^5-190.q^6-128.q^7-65.q^8-18.q^9-1.q^{10}}{2.q^4+12.q^5+2.q^6}$	13
6	$\frac{1.q^0+5.q^1+17.q^2+36.q^3+46.q^4+46.q^5+46.q^6+36.q^7+17.q^8+5.q^9+1.q^{10}}{2.q^4+2.q^6}$	21
7	$\frac{7.q^0+26.q^1+75.q^2+152.q^3+174.q^4+156.q^5+174.q^6+152.q^7+75.q^8+26.q^9+7.q^{10}}{2.q^3+12.q^4+4.q^5+12.q^6+2.q^7}$	23
8	$\frac{-1.q^0-8.q^1-20.q^2-24.q^3-22.q^4-24.q^5-20.q^6-8.q^7-1.q^8}{2.q^3+2.q^5}$	32
9	$\frac{-22.q^0-92.q^1-170.q^2-200.q^3-170.q^4-92.q^5-22.q^6}{2.q^2+12.q^3+2.q^4}$	121
10	$\frac{2.q^0+2.q^1+12.q^2+14.q^3+4.q^4+14.q^5+12.q^6+2.q^7+2.q^8}{2.q^3+2.q^5}$	132
11	$\frac{-2.q^0-12.q^1-40.q^2-52.q^3-44.q^4-52.q^5-40.q^6-12.q^7-2.q^8}{2.q^3+12.q^4+2.q^5}$	212
12	$\frac{-1.q^0-2.q^1-12.q^2-14.q^3-6.q^4-14.q^5-12.q^6-2.q^7-1.q^8}{2.q^3+2.q^5}$	213
13	$\frac{1.q^0+22.q^1+88.q^2+170.q^3+206.q^4+170.q^5+88.q^6+22.q^7+1.q^8}{2.q^3+12.q^4+2.q^5}$	232
14	$\frac{6.q^0+8.q^1+4.q^2+8.q^3+6.q^4}{2.q^2}$	323
15	$\frac{3.q^0+6.q^1+5.q^2+4.q^3+5.q^4+6.q^5+3.q^6}{2.q^2+2.q^4}$	1212
16	$\frac{12.q^0+32.q^1+40.q^2+32.q^3+12.q^4}{2.q^1+12.q^2+2.q^3}$	1213
17	$\frac{-3.q^0-2.q^1-5.q^2-12.q^3-5.q^4-2.q^5-3.q^6}{2.q^2+2.q^4}$	1232
18	$\frac{1.q^0+4.q^1+11.q^2+16.q^3+11.q^4+4.q^5+1.q^6}{2.q^3}$	1321
19	$\frac{8.q^0+12.q^1+24.q^2+40.q^3+24.q^4+12.q^5+8.q^6}{2.q^2+12.q^3+2.q^4}$	1323
20	$\frac{-6.q^0-8.q^1-4.q^2-8.q^3-6.q^4}{2.q^1+2.q^3}$	2121
21	$\frac{-5.q^0-4.q^1-44.q^2-60.q^3-30.q^4-60.q^5-44.q^6-4.q^7-5.q^8}{2.q^2+12.q^3+4.q^4+12.q^5+2.q^6}$	2123
22	$\frac{-1.q^0-5.q^1-11.q^2-14.q^3-11.q^4-5.q^5-1.q^6}{2.q^3}$	2321
23	$\frac{-3.q^0-6.q^1-5.q^2-4.q^3-5.q^4-6.q^5-3.q^6}{2.q^2+2.q^4}$	2323
24	$\frac{2.q^0+4.q^1+4.q^2+4.q^3+2.q^4}{2.q^2}$	3212
25	$\frac{-1.q^0-4.q^1-6.q^2-4.q^3-1.q^4}{2.q^2}$	3213
26	$\frac{6.q^0+8.q^1+4.q^2+8.q^3+6.q^4}{2.q^1+2.q^3}$	3232
27	$\frac{16.q^0+32.q^1+16.q^2}{2.q^0+12.q^1+2.q^2}$	12121
28	$\frac{4.q^0+8.q^1+40.q^2+8.q^3+4.q^4}{2.q^1+12.q^2+2.q^3}$	12123
29	$\frac{-3.q^0-8.q^1-4.q^2-8.q^3+46.q^4-8.q^5-4.q^6-8.q^7-3.q^8}{2.q^2+12.q^3+4.q^4+12.q^5+2.q^6}$	12132
30	$\frac{-8.q^0}{2.q^0}$	12321

Figure 21: A relation in $\mathcal{B}_4^H$ from GCT4

$Number$	$Coefficient$	σ
31	$\frac{-4.q^0-8.q^1-40.q^2-8.q^3-4.q^4}{2.q^1+12.q^2+2.q^3}$	12323
32	$\frac{-3.q^0-4.q^1-2.q^2-4.q^3-3.q^4}{2.q^1+2.q^3}$	13212
33	$\frac{-9.q^0-6.q^1-55.q^2+12.q^3-55.q^4-6.q^5-9.q^6}{2.q^1+12.q^2+4.q^3+12.q^4+2.q^5}$	13232
34	$\frac{9.q^0+6.q^1+55.q^2-12.q^3+55.q^4+6.q^5+9.q^6}{2.q^1+12.q^2+4.q^3+12.q^4+2.q^5}$	21213
35	$\frac{1.q^0-1.q^1+3.q^2-6.q^3+3.q^4-1.q^5+1.q^6}{2.q^2+2.q^4}$	21232
36	$\frac{-1.q^0+2.q^2-1.q^4}{2.q^2}$	21321
37	$\frac{2.q^0+3.q^1+6.q^2-3.q^3-16.q^4-3.q^5+6.q^6+3.q^7+2.q^8}{2.q^2+12.q^3+4.q^4+12.q^5+2.q^6}$	21323
38	$\frac{3.q^0+4.q^1+2.q^2+4.q^3+3.q^4}{2.q^1+2.q^3}$	23213
39	$\frac{-16.q^0-32.q^1-16.q^2}{2.q^0+12.q^1+2.q^2}$	23232
40	$\frac{3.q^0+4.q^1+2.q^2+4.q^3+3.q^4}{2.q^1+2.q^3}$	32121
41	$\frac{8.q^0}{2.q^0}$	32123
42	$\frac{1.q^0-2.q^2+1.q^4}{2.q^2}$	32132
43	$\frac{-3.q^0-4.q^1-2.q^2-4.q^3-3.q^4}{2.q^1+2.q^3}$	32321
44	$\frac{-8.q^0-16.q^1-8.q^2}{2.q^0+12.q^1+2.q^2}$	121213
45	$\frac{-1.q^0-14.q^1-15.q^2-4.q^3-15.q^4-14.q^5-1.q^6}{2.q^1+12.q^2+4.q^3+12.q^4+2.q^5}$	121232
46	$\frac{-2.q^0-4.q^1-2.q^2}{2.q^1}$	121321
47	$\frac{-2.q^0+4.q^2-2.q^4}{2.q^1+12.q^2+2.q^3}$	123213
48	$\frac{8.q^0+16.q^1+8.q^2}{2.q^0+12.q^1+2.q^2}$	123232
49	$\frac{-1.q^0-2.q^1-1.q^2}{2.q^1}$	132121
50	$\frac{2.q^0-4.q^2+2.q^4}{2.q^1+12.q^2+2.q^3}$	132123
51	$\frac{2.q^0+8.q^1+12.q^2+8.q^3+2.q^4}{2.q^1+12.q^2+2.q^3}$	212132
52	$\frac{2.q^0+4.q^1+2.q^2}{2.q^1}$	212321
53	$\frac{1.q^0+14.q^1+15.q^2+4.q^3+15.q^4+14.q^5+1.q^6}{2.q^1+12.q^2+4.q^3+12.q^4+2.q^5}$	212323
54	$\frac{3.q^0+8.q^1+10.q^2+8.q^3+3.q^4}{2.q^1+12.q^2+2.q^3}$	213212
55	$\frac{-2.q^0-8.q^1-12.q^2-8.q^3-2.q^4}{2.q^1+12.q^2+2.q^3}$	213232
56	$\frac{-1.q^0-2.q^1-1.q^2}{2.q^1}$	232121
57	$\frac{-3.q^0-8.q^1-10.q^2-8.q^3-3.q^4}{2.q^1+12.q^2+2.q^3}$	232132
58	$\frac{1.q^0+2.q^1+1.q^2}{2.q^1}$	232321
59	$\frac{-2.q^0-4.q^1-2.q^2}{2.q^1}$	321232
60	$\frac{2.q^0+4.q^1+2.q^2}{2.q^1}$	321323

Figure 22: A relation in $\mathcal{B}_4^H$ from GCT4 continued.

$Number$	$Coefficient$	σ
61	$\frac{1.q^0+2.q^1+1.q^2}{2.q^1}$	323212
62	$\frac{1.q^0-2.q^1+1.q^2}{2.q^0+2.q^2}$	1212132
63	$\frac{2.q^0}{2.q^0}$	1213213
64	$\frac{1.q^0-2.q^1+1.q^2}{2.q^0+2.q^2}$	1213232
65	$\frac{2.q^0}{2.q^0}$	1232121
66	$\frac{2.q^0-4.q^1+2.q^2}{2.q^0+12.q^1+2.q^2}$	1232132
67	$\frac{16.q^1}{2.q^0+12.q^1+2.q^2}$	1321232
68	$\frac{-2.q^0}{2.q^0}$	1321323
69	$\frac{-1.q^0+2.q^1-1.q^2}{2.q^0+2.q^2}$	2121323
70	$\frac{-4.q^0-8.q^1-4.q^2}{2.q^0+12.q^1+2.q^2}$	2123212
71	$\frac{-16.q^1}{2.q^0+12.q^1+2.q^2}$	2123213
72	$\frac{-1.q^0+2.q^1-1.q^2}{2.q^0+2.q^2}$	2123232
73	$\frac{-2.q^0+4.q^1-2.q^2}{2.q^0+12.q^1+2.q^2}$	2132123
74	$\frac{4.q^0+8.q^1+4.q^2}{2.q^0+12.q^1+2.q^2}$	2321232

Figure 23: A relation in $\mathcal{B}_4^H$ from GCT4 continued.